Two Generalizations of the Wedderburn-Artin Theorem with Applications [*][†][‡]

M. Behboodi[a,b][§] **A. Daneshvar**[a] **and M. R. Vedadi**[a]

[a]Department of Mathematical Sciences, Isfahan University of Technology
P.O.Box : 84156-83111, Isfahan, Iran

[b]School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O.Box : 19395-5746, Tehran, Iran
mbehbood@cc.iut.ac.ir
a.daneshvar@math.iut.ac.ir
mrvedadi@cc.iut.ac.ir

Abstract

We say that an R-module M is *virtually simple* if $M \neq (0)$ and $N \cong M$ for every non-zero submodule N of M, and *virtually semisimple* if each submodule of M is isomorphic to a direct summand of M. We carry out a study of virtually semisimple modules and modules which are direct sums of virtually simple modules. Our theory provides two natural generalizations of the Wedderburn-Artin Theorem and an analogous to the classical Krull-Schmidt Theorem. Some applications of these theorems are indicated. For instance, it is shown that the following statements are equivalent for a ring R: (i) Every finitely generated left (right) R-modules is virtually semisimple; (ii) Every finitely generated left (right) R-modules is a direct sum of virtually simple modules; (iii) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k, n_1, \ldots, n_k \in \mathbb{N}$ and each D_i is a principal ideal V-domain; and (iv) Every non-zero finitely generated left R-module can be written uniquely (up to isomorphism and order of the factors) in the form $Rm_1 \oplus \ldots \oplus Rm_k$ where each Rm_i is either a simple R-module or a left virtually simple direct summand of R.

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1   Introduction

The subject of determining structure of rings and algebras over which all (finitely gener-
ated) modules are direct sums of certain cyclic modules has a long history. One of the
first important contributions in this direction is due to Wedderburn [27]. He showed that
every module over a finite-dimensional K-algebra A is a direct sum of simple modules if
and only if $A \cong \prod_{i=1}^{m} M_{n_i}(D_i)$ where $m, n_1, \dots, n_m \in \mathbb{N}$ and each D_i is finite-dimensional
division algebra over K. After that in 1927, E. Artin generalizes the Wedderburn's theo-
rem for semisimple algebras ([2]). Wedderburn-Artin's result is a landmark in the theory
of non-commutative rings. We recall this theorem as follows:

Theorem 1.1. (Wedderburn-Artin Theorem). *For a ring R, the following conditions are
equivalent:*

(1) *Every left (right) R-module is a direct sum of simple modules.*
(2) *Every finitely generated left (right) R-module is a direct sum of simple modules.*
(3) *The left (right) R-module R is a direct sum of simple modules.*
(4) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ *where $k, n_1, \dots, n_k \in \mathbb{N}$ and each D_i is a division ring.*

Another one of the important contributions in this direction is due to G. Köthe [22].
He considered rings over which all modules are direct sums of cyclic modules. Köthe in
[22] proved the following. We recall that an *Artinian* (resp., *Noetherian*) *ring* is a ring
which is both a left and right Artinian (resp., Noetherian). A *principal ideal ring* is a ring
which is both a left and a right principal ideal ring.

Theorem 1.2. (Köthe). *Over an Artinian principal ideal ring, each module is a direct
sum of cyclic modules. Furthermore, if a commutative Artinian ring has the property that
all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal
ring.*

Later Cohen and Kaplansky [9] obtained the following result:

Theorem 1.3. (Cohen and Kaplansky). *If R is a commutative ring such that each R-
module is a direct sum of cyclic modules, then R must be an Artinian principal ideal ring.*

However, finding the structure of non-commutative rings each of whose modules is a
direct sum of cyclic modules is still an open question; see [26, Appendix B, Problem 2.48]
and [18, Question 15.8] (for a partial solution, we refer [6]). Further, Nakayama in [25,
Page 289] gave an example of a non-commutative right Artinian ring R where each right
R-module is a direct um of cyclic modules but R is not a principal right ideal ring.

Also, the problem of characterizing rings over which all finitely generated modules are direct sums of cyclic modules (called FGC-rings) was first raised by I. Kaplansky [20], [21] for the commutative setting. The complete characterization of commutative FGC rings is a deep result that was achieved in the 1970s. A paper by R. Wiegand and S. M. Wiegand [28] and W. Brandal's book [7] are two sources from which to learn about this characterization. The corresponding problem in the non-commutative case is still open; see [26, Appendix B. Problem 2.45] (for a partial solution, we refer [3] and [4]).

In this paper we say that an R-module M is *virtually simple* if $M \neq (0)$ and $N \cong M$ for every nonzero submodule N of M (i.e., up to isomorphism, M is the only non-zero submodule of M). Clearly, we have the following implications for ${}_R M$:

$$M \text{ is simple} \Rightarrow M \text{ is virtually simple} \Rightarrow M \text{ is cyclic}$$

Note that these implications are irreversible in general when R is not a division ring.

The above considerations motivated us to study rings for which every (finitely generated) module is a direct sum of virtually simple modules. Since any injective virtually simple module is simple, so each left R-module is a direct sum of virtually simple modules if and only if R is semisimple (see Proposition 2.2 and Corollary 2.3). Now the following three interesting natural questions arise:

Question 1.4. *Describe rings R where each finitely generated left R-module is a direct sum of virtually simple modules.*

Question 1.5. *Describe rings R where the left R-module R is a direct sum of virtually simple modules.*

Question 1.6. *Whether the Krull-Schmidt Theorem holds for direct sums of virtually simple modules?*

One goal of this paper is to answer the above questions.

We note that a semisimple module is a type of module that can be understood easily from its parts. More precisely, a module M is semisimple if and only if every submodule of M is a direct summand. This property motivates us to study modules for which every submodule is isomorphic to a direct summand. In fact, the notions of "virtually semisimple modules" and "completely virtually semisimple modules" were introduced and studied in our recent work [5] as generalizations of semisimple modules. We recall that an R-module M is *virtually semisimple* if each submodule of M is isomorphic to a direct summand of M. If each submodule of M is a virtually semisimple module, we call M *completely virtually semisimple.* We also have the following implications for ${}_R M$:

M is semisimple $\Rightarrow$ M is completely virtually semisimple $\Rightarrow$ M is virtually semisimple

These implications are also irreversible in general (see [5, Examples 3.7 and 3.8]).

If $_RR$ (resp., R_R) is a virtually semisimple module, we then say that R is a *left* (resp., *right*) *virtually semisimple ring*. A *left* (resp., *right*) *completely virtually semisimple ring* is similarly defined (these notions are not left-right symmetric). In [5, Theorems 3.4 and 3.13], we gave several characterizations of left (completely) virtually semisimple rings.

Clearly, an R-module M is virtually simple if and only if M is a non-zero indecomposable virtually semisimple module. We note that a semisimple module is a direct sum (finite or not) of simple modules, but it is not true when we replace "semisimple" by "virtually semisimple" and "simple" by "virtually simple" (see Example 3.2). It is not hard to show that if every left R-module is virtually semisimple, then R is a semisimple ring. Nevertheless, the Wedderburn-Artin theorem motivates us to study rings for which every finitely generated left (right) module is a virtually semisimple module. In fact, the following interesting natural questions arise:

Question 1.7. *Describe rings where each finitely generated left R-module is completely virtually semisimple.*

Question 1.8. *Describe rings where each finitely generated left R-module is virtually semisimple.*

Question 1.9. *Describe rings where each cyclic left R-module is virtually semisimple.*

Therefore, the second goal of this paper is to answer the above questions, however, Question 1.9 remains open to discussion in the non-commutative case.

In Section 2, we give two generalizations of the Wedderburn-Artin Theorem (Theorems 2.7 and 2.14). Also, we prove a unique decomposition theorem for finite direct sum of virtually simple modules, which is an analogous to the classical Krull-Schmidt Theorem (Theorem 2.17). Section 3 consists of some applications of these theorems. Our version of the Krull-Schmidt Theorem applies to prove that every finitely generated complectly virtually semisimple module can be written "uniquely" as a direct sum of virtually simple modules (see Proposition 3.1). Finally, as an important application, we give a structure theorem for rings whose finitely generated left (right) R-modules are direct sums of virtually simple modules (Proposition 3.3 and Theorem 3.4).

Throughout this paper, all rings are associative with identity and all modules are unitary. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1, 15, 23, 29].

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2 Generalizations of Wedderburn-Artin and Krull-Schmidt Theorems

Let M and N be two R-modules. The notation $N \leq M$ (resp., $N \leq_e M$) means that N is a submodule (resp., an essential submodule). We use the notation $M \hookrightarrow N$ to denote that M embeds in N. An *essential monomorphism*, denoted by $M \overset{ess}{\hookrightarrow} N$, from M to N is any monomorphism $f : M \longrightarrow N$ such that $f(M) \leq_e N$. Also, we use the notation $E(_R M)$ for the *injective hull* of M.

Following [15], we denote by u.dim(M) and K.dim(M) the *uniform dimension* and *Krull dimension* of a module M, respectively. If $\alpha \geq 0$ is an ordinal number then the module M is said to be α-critical provided K.dim$(M) = \alpha$ while K.dim$(M/N) < \alpha$ for all non-zero submodules N of M. A module is called *critical* if it is α-critical for some ordinal $\alpha \geq 0$. It is known that critical modules are uniform (see [24, Lemma 6.2.12]). We say that a left ideal P of a ring R is *quasi-prime* if $P \neq R$ and, for ideals $A, B \subseteq R$, $AB \subseteq P \subseteq A \cap B$ implies that $A \subseteq P$ or $B \subseteq P$.

The following result is very useful in our investigation.

Lemma 2.1. (See [5, Proposition 2.7]). *Let M be a non-zero virtually semisimple left R-module. Then;*

(i) *The following conditions are equivalent.*

    (1) u.dim$(M) < \infty$.

    (2) M *is finitely generated.*

    (3) $M \cong R/P_1 \oplus \ldots \oplus R/P_n$ *where $n \in \mathbb{N}$ and each P_i is a quasi-prime left ideal of R such that R/P_i is a critical Noetherian R-modules.*

(ii) *If M is finitely generated, then $M \cong N$ for all $N \leq_e M$.*

Proposition 2.2. *Every quasi-injective virtually semisimple module M is semisimple.*

Proof. Assume that $N \leq M$. By the assumption, $M = K \oplus L$ where $K \cong N$ and $K, L \leq M$. Since K is a direct summand of M, so K is M-injective and so N is M-injective. It follows that N is a direct summand of M. Thus M is a semisimple module. $\qquad\square$

Corollary 2.3. *The following conditions are equivalent for a ring R.*

(1) *Every left (right) R-module is a direct sum of virtually simple module.*

(2) *Every left (right) R-module is virtually semisimple.*

(3) R *is a semisimple ring.*

Proof. (1) $\Rightarrow$ (3). By assumption, $E(_R R)$ is a direct sum of injective virtually simple R-module. Since every injective module is quasi-injective, so by Proposition 2.2, $E(_R R)$ is a semisimple R-module and hence R is semisimple.

(2) $\Rightarrow$ (3) can be proven by a similar way.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are evident. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We recall that the *singular submodule* $Z(M)$ of a left (resp., right) R-module M consisting of elements whose annihilators are essential left (resp., right) ideals in R. An R-module M is called a *singular* (resp., *non-singular*) *module* if $Z(M) = M$ (resp., $Z(M) = 0$).

Lemma 2.4. *Let R_1 and R_2 be rings and $T = R_1 \oplus R_2$. Let "P" denote any one of the properties: finitely generated, singular, non-singular, projective, injective, semisimple, virtually semisimple and virtually simple. Then by the natural multiplication every left T-module M has the form $M_1 \oplus M_2$ where each M_i is a left R_i-module and the T-module M satisfies the property "P" if and only if each R_i-module M_i satisfies the property.*

Proof. For the first part, we just note that T has two central orthogonal idempotent elements e_1 and e_2 with $e_1 + e_2 = 1_T$ and $Te_i = R_i$. Thus if M is a left T-module then $M = e_1 M \oplus e_2 M$ where each $e_i M$ is a left R_i-module. Set $M_i = e_i M$ ($i = 1, 2$). In this situation any submodule of M_i has the form $e_i K$ for some $K \leq {}_T M$. Thus the proof in the injectivity case is easily obtained by Baer injective test. For the other cases there are routine arguments by using $\text{Soc}(_T M) = \text{Soc}(_{R_1} M_1) \oplus \text{Soc}(_{R_2} M_2)$, $Z(_T M) = Z(_{R_1} M_1) \oplus Z(_{R_2} M_2)$ and by the fact that if $X \oplus Y \cong M$ is an isomorphism of left T-modules then $e_i X \oplus e_i Y \cong e_i M$ is an isomorphism of left R_i-modules. $\qquad\qquad$ $\square$

Lemma 2.5. *Let M and N be virtually simple R-modules. Then;*

(i) *M is a cyclic critical Noetherian uniform R-module.*

(ii) *If M and N are virtually simple R-modules with $\text{Hom}_R(M, N) = 0$ then $M \cong N$ or $Z(N) = N$.*

(iii) *If $M \not\cong N$ and N is projective, then $\text{Hom}_R(M, N) = 0$.*

Proof. (i) Let M be a virtually semisimple module. Clearly M is cyclic and hence $M \cong R/P_i$ for some i as stated in Lemma 2.1(i). The last statement is now true because critical modules are uniform.

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(ii) Suppose that $_RM$ and $_RN$ are virtually simple and $0 \neq f \in \mathrm{Hom}_R(M, N)$. We have $M/\mathrm{Ker} f \cong \mathrm{Im} f \cong N$. Now if $\mathrm{Ker} f = 0$, then $M \cong N$ and if $\mathrm{Ker} f \neq 0$, then $\mathrm{Ker} f \leq_e M$ because $_RM$ is uniform by (i). Hence N must be singular.

(iii) By (ii) and the fact that projective modules are not singular. $\qquad \square$

Next we have the following lemma.

Lemma 2.6. *Let M be a projective virtually simple R-module. Then the endomorphism ring $\mathrm{End}_R(M)$ is a principal left ideal domain.*

Proof. This follows from [16, Corollary 2.8] and [5, Theorem 2.9]. $\qquad \square$

Being a left virtually semisimple ring is not Morita invariant (see [5, Example 3.8]). Surprisingly, being a left completely virtually semisimple ring is a Morita invariant property (see [5, Proposition 3.3]). In addition, being (completely) virtually semisimple module is a Morita invariant property (see [5, Proposition 2.1 (iv)]). For an R-module M and each $n \in \mathbb{N}$, we use the notation $M^{(n)}$ instead of $M \oplus \cdots \oplus M$ (n times).

We are now in a position to give the following generalization of the Wedderburn-Arttin Theorem. We remark that the equivalences between (2) and (3) below has been shown in [5, Theorem 3.13].

Theorem 2.7. (First generalization of the Wedderburn-Artin Theorem) *The following statements are equivalent for a ring R.*

(1) The left R-module R is a direct sum of virtually simple modules.

(2) R is a left completely virtually semisimple ring.

(3) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k, n_1, ..., n_k \in \mathbb{N}$ and each D_i is a principal left ideal domain.

Moreover, in the statement (3), the integers k, $n_1, ..., n_k$ and the principal left ideal domains $D_1, ..., D_k$ are uniquely determined (up to isomorphism) by R.

Proof. (1) $\Rightarrow$ (3). By assumption, $R = I_1 \oplus \cdots \oplus I_n$ where $n \in \mathbb{N}$ and each I_i is a (projective) virtually simple R-module. Grouping these according to their isomorphism types as left R-modules, so we can assume that $R = I_1^{(n_1)} \oplus \cdots \oplus I_k^{(n_k)}$ where $1 \leq k \leq n$ and $I_i \not\cong I_j$ for any pair $i \neq j$. Thus, $R \cong \mathrm{End}_R(R) = \mathrm{End}_R\big(I_1^{(n_1)} \oplus \cdots \oplus I_k^{(n_k)}\big)$. Also, by Lemma 2.5(iii), we have $\mathrm{Hom}_R(I_s, I_t) = 0$ for every $s \neq t$. It follows that $R \cong \bigoplus_{i=1}^{k} \mathrm{End}_R\big(I_i^{(n_i)}\big) \cong \bigoplus_{i=1}^{k} M_{n_i}\big(\mathrm{End}_R(I_i)\big)$. Now By Lemma 2.6, the endomorphism ring $D_j := \mathrm{End}_R(I_j)$ is a principal left ideal domain for each $1 \leq j \leq k$. This shows that $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$, and the proof is complete.

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$(3) \Rightarrow (1)$. By Lemma 2.4, we can assume that $k = 1$, i.e., $R = M_n(D)$ where $n \in \mathbb{N}$ and D is a principal ideal domain. It is known that R is Morita equivalent to D (see for instance [23, Page 525]). Let $D \overset{\mathcal{F}}{\approx} R$. Then $\mathcal{F}(D) = D^{(n)}$ is a virtually semisimple R-module because $_DD$ is virtually semisimple. We set

$$N_j = \begin{pmatrix} 0 & \cdots & 0 & \overset{j-\text{th}}{D} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & D & 0 & \cdots & 0 \end{pmatrix} \qquad (1 \leq j \leq n).$$

Then by the natural matrix multiplication, N_j is a left R-module with $N_j \cong D^{(n)}$. Thus each N_j is virtually simple, i.e., R is a direct sum of virtually simple left R-modules.

$(2) \Leftrightarrow (3)$ and the *"moreover statement"* are by [5, Theorem 3.13]. $\square$

A ring R is called a *left* (resp., *right*) V-*ring* if each simple left (resp., right) R-module is injective. We say that R is V-*ring* if it is both left and right V-ring.

Remark 2.8. Although there exists an example of a non-domain which is a left V-ring but not a right V-ring, the question whether a left (right) V-domain is a right (left) V-domain remains open in general. See [19, Corollary 3.3] where the authors proved that the answer is positive for principal ideal domains.

We need the following proposition.

Proposition 2.9. *Let $R = \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k, n_1, \ldots, n_k \in \mathbb{N}$ and each D_i is a principal ideal V-domain. Then every finitely generated left R-module is a direct sum of a projective module and a singular (injective) semisimple module.*

Proof. By Lemma 2.4, we can assume that $k = 1$, i.e., $R = M_n(D)$ where $n \in \mathbb{N}$ and D is a principal ideal V-domain. It is well-known that properties of being projective, being injective, being finitely generated and being singular are Morita invariant. Since D is Morita equivalent to $M_n(D)$, so we can assume that $n = 1$, i.e., $R = D$. Let M be a finitely generated left D-module. By [10, Theorem 1.4.10], $M \cong D/I_1 \oplus \cdots D/I_n \oplus D^{(m)}$ for some non-zero left ideals I_i $(1 \leq i \leq n)$ of D and $m \in \mathbb{N} \cup \{0\}$. Since every non-zero left ideal of D is essential, so each D/I_i is singular. It follows that $M = Z(M) \oplus P$ where P is a projective left D-module.

Now let S be any cyclic D-submodule in $Z(M)$. Since D is a hereditary Noetherian ring, by [24, Proposition 5.4.6], S has a finite length. It follows that $\text{Soc}(Z(M)) \leq_e Z(M)$.

Also since D is Noetherian, $Z(M)$ and so $\text{Soc}(Z(M))$ is finitely generated. Thus the V-domain condition on D implies that $\text{Soc}(Z(M))$ must be a direct summand of $Z(M)$, proving that $\text{Soc}(Z(M)) = Z(M)$. Therefore, $M = Z(M) \oplus P$ where $Z(M)$ is a semisimple (injective) module and P is a projective module, as desired. □

We are now going to give the following another generalization of the Wedderburn-Artin Theorem.

Let R be a ring and M an R-module. We recall that a submodule N of M is (*essentially*) *closed* if $N \leq_e K \leq M$ always implies $N = K$. Also, the module M is called *extending* (or *CS-module*) if every closed submodule of M is a direct summand of M. Given $n \in \mathbb{N}$, a uniform R-module U is called an n-CS^+ module if $U^{(n)}$ is extending and each uniform direct summand of $U^{(n)}$ is isomorphic to $_R U$. An integral domain in which every finitely generated left ideal is principal is called a *left Bezout domain*. *Right Bezout* domains are defined similarly, and when both conditions hold we speak of a Bezout domain.

The following lemmas are needed.

Lemma 2.10. (See [12, Theorem 2.2]) *Let R be a simple ring. Then R contains a uniform left ideal U such that $_R U$ is 2-CS^+ if and only if R is isomorphic to the $k \times k$ matrix ring over a Bezout domain D for some $k \in \mathbb{N}$.*

Lemma 2.11. ([10, Proposition 2.3.17]) *If R is a right Bezout domain then R is right Ore domain.*

Lemma 2.12. (See [8, Lemma 1]) *Let R be a semihereditary and Goldie ring with classical quotient ring Q. Let S be a simple right R-module. Then S is finitely presented if and only if S may be embedded in the module $(Q/R) \oplus R$.*

Lemma 2.13. (See [8, Theorem 2]) *Let R be a left Noetherian, left hereditary, semiprime right Goldie with classical quotient ring Q. Then R is right Noetherian if and only every simple right R-module can be embedded in $(Q/R) \oplus R$.*

Theorem 2.14. (Second generalization of the Wedderburn-Artin Theorem). *The following statements are equivalent for a ring R.*

(1) *All finitely generated left R-modules are virtually semisimple.*

(1′) *All finitely generated right R-modules are virtually semisimple.*

(2) *All finitely generated left R-modules are completely virtually semisimple.*

(2′) *All finitely generated right R-modules are completely virtually semisimple.*

(3) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ *where each D_i is a principal ideal V-domain.*

Proof. Since the statement (3) is symmetric, we only need to prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) is by Lemma 2.1 (not that every finitely generated virtually semisimple module is Noetherian).

(2) $\Rightarrow$ (1) is evident.

(2) $\Rightarrow$ (3). By assumption, R is left completely virtually semisimple and so by Theorem 2.7, $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where $k \in \mathbb{N}$ and each D_i is a principal left ideal domain. By Remark 2.8, it suffices to prove that each D_i is a principal right ideal domain and a left V-domain. Let $D = D_i$ for some i. By (2) and the fact that "completely virtually semisimplity is a Morita invariant property for modules", we deduce that all finitely generated left D-modules are also completely virtually semisimple.

Now let S be a simple D-module. Assume that $E = E(S)$ is the injective hull of $_D S$ and C is a cyclic D-submodule of E. Since $S \leq_e E$, we have $S \leq_e C$ and by Lemma 2.1(ii), $C \cong S$. It follows that $E = S$ and hence D is a left V-domain.

We claim that the left D-module $D \oplus D$ is extending. To see this, assume that N is a closed D-submodule of $D \oplus D$. If u.dim$(N) = 2$ then N is an essential submodule of $D \oplus D$ and hence $N = D \oplus D$. If u.dim$(N) = 1$, then by [23, Theorem 6.37], u.dim$((D \oplus D)/N) = 1$. Set $U = (D \oplus D)/N$. Then by assumption, U is a finitely generated uniform left virtually semisimple D-module and so $U \cong D/P$ where P is a left ideal of D by Lemma 2.1(i). Now if $Z(U) = K/N$ where $N \leq K \leq D \oplus D$, then by Kaplansky's Theorem [23, Theorem 2.24], K is a free (projective) left D-module. The singularity of K/N implies that $N \leq_e K$. Since N is closed, we have $N = K$ and hence U is a non-singular left D-module. It follows that $P = 0$ (because D is a principal left ideal domain and every non-zero left ideal in D is essential). Thus $U \cong D$ and so $_D U$ is projective. This shows that N is a direct summand of $D \oplus D$. Therefore, $D \oplus D$ is a left extending D-module, as desired.

It is now clear that D is a left 2-CS$^+$ D-module. Also since D is a left V-domain, it is a simple ring. Thus by Lemma 2.10, D is a right Bezout domain. To complete the proof it now remains to prove that D is a right Noetherian ring.

By Lemma 2.11, D is a right Ore domain. Since D is a left hereditary ring, by [13, Corollary 12.18] D is a right semihereditary ring. We are applying Lemma 2.13 to show that D is right Noetherian. Now assume that S is a right simple D-module. If $S \cong D$ then D is semisimple and we are done. Thus we can assume $S \cong D/P$ where P is a non-zero maximal right ideal D. Since D is right Ore, so P is a right essential ideal of D and hence S_D is torsion. Thus by [10, Proposition 5.3.6], S_D is finitely presented. It follows that S can be embedded in the right D-module $(Q/D) \oplus D$ by Lemma 2.12 where Q is the classical quotient ring of D, and the proof is complete.

$(3) \Rightarrow (1)$. By Lemma 2.4, we can assume that $k = 1$, i.e., $R = M_n(D)$ where $n \in \mathbb{N}$ and D is a principal ideal V-domain. Since being completely virtually semisimple is Morita invariant property, it is enough to show that every finitely generated left D-module is completely virtually semisimple. Assume that M is a finitely generated left D-module. By Proposition 2.9, $M = S \oplus P$ where S is a semisimple D-module and P is a projective D-module. Thus by [5, Propositions 3.3 (i)], P is virtually semisimple. We can assume that $\mathrm{Soc(D)} = 0$ (otherwise D is a division ring and we are done), so we can deduce that $\mathrm{Soc}(P) = 0$ and by [5, Propositions 2.3 (ii)], M is virtually semisimple and the proof is complete. $\square$

The following example, originally from Cozzens [11], shows that there are principal ideal V-domains which are not division rings.

Example 2.15. ([17, Example of Page 46]) Let K be a universal differential field with derivation d and let $D = K[y; d]$ denote the ring of differential polynomials in the indeterminate y with coefficients in K, i.e., the additive group of $K[y; d]$ is the additive group of the ring of polynomials in the indeterminate y with coefficients in field K, and multiplication in D is defined by: $ya = ay + d(a)$ for all a in K. It is shown that D is both left and right principal ideal domain, the simple left D-modules are precisely of the form $V_a = D/D(y - a)$ where a in K and each simple left D-module is injective . Hence D is a left V-ring. Similarly, D is a right V-ring.

In the following, we obtain a uniqueness decomposition theorem for finite direct sum of virtually simple modules, which is analogous to the classical Krull-Schmidt Theorem for direct sum decompositions of modules.

Lemma 2.16. *Let $M = V_1 \oplus \cdots \oplus V_n$ be a direct sum of virtually simple left R-modules. Then:*

(i) *If $N \leq_e {}_R M$ then $M \hookrightarrow N$.*

(ii) *If $0 \neq N \leq {}_R M$ then there is an index j such that $V_j \hookrightarrow N$.*

Proof. (i) Assume that $N \leq_e M$. Then $N \cap V_i \neq 0$ and so $N \cap V_i \cong V_i$ for each i. This shows that $M \cong \bigoplus_{i=1}^n (N \cap V_i) \subseteq N$.

(ii) Assume that $0 \neq N \leq M$. We can prove that the result by induction on n. Just consider the cases $N \cap V_i \neq 0$ or $N \cap V_1 \neq 0$. $\square$

Let M and N be R-modules. We say that M and N are *R-subisomorphic* if $M \hookrightarrow N$ and $N \hookrightarrow M$.

Theorem 2.17. (The Krull-Schmidt Theorem for virtually simple modules). *Let $M = V_1 \oplus \cdots \oplus V_n$ and $N = U_1 \oplus \cdots \oplus U_m$ where all V_i's and U_j's are virtually simple modules. If M and N are R-subisomorphic, then $n = m$ and there is a permutation σ on $\{1, ..., n\}$ such that $U_i \cong V_{\sigma(i)}$.*

Proof. Since virtually simple modules are uniform (Lemma 2.5(i)), so by our assumption, $m = \mathrm{u.dim}(N) \leq \mathrm{u.dim}(M) = n$ and vice versa. Thus $m = n$. Without loss of generality, we can assume that $M = X_1 \oplus \cdots \oplus X_l$ and $N = Y_1 \oplus \cdots \oplus Y_t$ where $X_i = V_i^{(m_i)}$ $(1 \leq i \leq l)$ with $V_i \not\cong V_s$ $(i \neq s)$ and $Y_j = U_j^{(n_j)}$ $(1 \leq j \leq t)$ with $U_j \not\cong U_k$ $(j \neq k)$. Note that if $V_i \hookrightarrow U_k$ and $X_i \cap \left(\Sigma_{j \neq k} Y_j \right) = W$, then $W = 0$. Otherwise, there is a nonzero embedding $W \hookrightarrow X_i$ and so by Lemma 2.16(ii), $V_i \hookrightarrow W$. It follows that $V_i \hookrightarrow \Sigma_{j \neq k} Y_j$ and hence $V_i \hookrightarrow U_h$ for some $h \neq k$, a contradiction. Thus we can conclude that for each $i \in \{1, \cdots, l\}$ there exists a unique $k \in \{1, \cdots, t\}$ such that $X_i \hookrightarrow Y_k$. Similarly, each Y_j can be embedded in only one X_i's. This shows that $l \leq t$, and $t \leq l$, i.e., $t = l$. Clearly $V_i \hookrightarrow U_k$ if and only if $V_i \cong U_k$. Thus it is enough to show that $m_i = n_k$ when $X_i \hookrightarrow Y_k$. Again consider that if $X_i \hookrightarrow Y_k$ and $Y_k \hookrightarrow X_s$, then Lemma 2.16 proves that $i = s$. This shows that $\mathrm{u.dim}(X_i) = m_i \leq \mathrm{u.dim}(Y_k) = n_k$ and vice versa, and hence the proof is now complete. $\square$

3   Some applications

We give a structure theorem for rings over which every finitely generated module is a direct sum of virtually simple modules. Such rings form a proper subclass of the class of FGC rings. As an application of Theorem 2.17, we first show that every completely virtually simple module is uniquely (up to isomorphism) a direct sum of virtually simple modules, but the converse is not true in general.

Proposition 3.1. *Every finitely generated completely virtually semisimple module is a direct sum of virtually simple modules. Up to a permutation, the virtually simple components in such a direct sum are uniquely determined up to isomorphism.*

Proof. Assume that M is a finitely generated complectly virtually semisimple module. By Theorem 2.17, it suffices to show that M is a finite direct sum of virtually simple modules. If M is virtually simple then we are done. Assume that M is not virtually simple. Thus there is a non-zero submodule N of M such that $M \not\cong N$. By assumption, $M = U \oplus W$ where $U \cong N$ and $0 \neq W \leq M$. If U and W are virtually simple then we are done. If not, without lose of generality, assume that U is not virtually simple. So U

has a non-zero submodule $N_1 \ncong U$. By assumption, U is again virtually semisimple and so $U = U_1 \oplus W_1$ such that $N_1 \cong U_1$ and $0 \neq W_1 \leq U$. It follows that $M = U_1 \oplus W_1 \oplus W$. If one of the U_1, W_1 or W is not virtually simple, for example U_1, then we may repeat the above argument with respect to U_1 and continue inductively. Since M is a finitely generated virtually semisimple module, u.dim$(M) < \infty$ by Lemma 2.1(i) and hence, we obtain virtually simple submodules $K_1, \ldots, K_n$ such that $M = \bigoplus_{i=1}^n K_i$, and the proof is completed. $\square$

Let R be a ring and M be a left R-module. If X is an element or a subset of M, we define the *annihilator* of X in R by $\mathrm{Ann}_R(X) = \{r \in R \mid rX = (0)\}$. In the case R is non-commutative and X is an element or a subset of an R, we define the *left annihilator* of X in R by $\mathrm{l.ann}_R(X) = \{r \in R \mid rX = (0)\}$ and the *right annihilator* of X in R by $\mathrm{r.ann}_R(X) = \{r \in R \mid Xr = (0)\}$.

The following example shows that the converse of Proposition 3.1 does not hold in general.

Example 3.2. Let F be a field and we set $R = F[[x, y]]/\langle xy \rangle$. It is clear that $\mathcal{M} = RX \oplus RY$ is a maximal ideal of R where $X = x + \langle xy \rangle$ and $Y = y + \langle xy \rangle$. It is easily see that $\mathrm{Spec}(R) = \{RX, RY, \mathcal{M}\}$. Consider $\mathcal{M}$ as an R-module and hence $\mathcal{M} \cong R/\mathrm{Ann_R}(X) \bigoplus R/\mathrm{Ann_R}(Y) = R/RY \bigoplus R/RX$. By [14, Theorem 2.1], the rings R/RX and R/RY are principal ideal domains because they have principal prime ideals. We show that ${}_R\mathcal{M}$ is not virtually semisimple. Note that $X^2 = X(X+Y)$ and $Y^2 = Y(X+Y)$ and so we have $RX^2 \oplus RY^2 \leq R(X+Y) \leq {}_R\mathcal{M}$. It follows that u.dim$({}_R\mathcal{M}) = $ u.dim$(R(X+Y))$ or equivalently $R(X+Y) \leq_e {}_R\mathcal{M}$. Now if ${}_R\mathcal{M}$ is virtually semisimple we must have $R \cong R(X+Y) \cong \mathcal{M}$, but $\mathcal{M}$ is not cyclic. Therefore ${}_R\mathcal{M}$ is not virtually semisimple.

The following result provides a plain structure for virtually simple modules over $M_n(D)$ where $n \in \mathbb{N}$ and D is a principal ideal V-domain.

Corollary 3.3. *Let $R \cong M_n(D)$ where $n \in \mathbb{N}$ and D is a principal ideal V-domain. Then a left R-module M is virtually simple if and only if*

$$M \cong \begin{pmatrix} D/P \\ D/P \\ \vdots \\ D/P \end{pmatrix}$$

where P is a maximal left ideal of D or $P = (0)$.

Proof. This is obtained by Proposition 2.9 and the familiar correspondence between modules over D and $M_n(D)$. $\qquad\square$

Theorem 3.4. *The following statements are equivalent for a ring R.*

(1) *Every finitely generated left R-modules is a direct sum of virtually simple modules.*

($1'$) *Every finitely generated right R-modules is a direct sum of virtually simple modules.*

(2) $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ *where $k, n_1, ..., n_k \in \mathbb{N}$ and each D_i is a principal ideal V-domain.*

(3) *Every finitely generated left R-modules is uniquely (up to isomorphism) a direct sum of cyclic left R-modules that are either simple or virtually simple direct summand of ${}_R R$.*

($3'$)*Every finitely generated right R-modules is uniquely (up to isomorphism) a direct sum of cyclic left R-modules that are either simple or virtually simple direct summand of ${}_R R$.*

(4) *Every finitely generated left R-module is an extending module that embeds in a direct sum of virtually simple modules.*

($4'$) *Every finitely generated right R-module is an extending module that embeds in a direct sum of virtually simple modules.*

Proof. Since the statement (2) is symmetric, we only need to prove $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

$(1) \Rightarrow (2)$. By Theorem 2.14, it suffices to prove that all finitely generated R-modules are virtually semisimple. By Theorem 2.7, R is left Noetherian. Let M be a finitely generated left R-module and $K \leq M$. It is well-known that $K \oplus L \leq_e M$ for some $L \leq M$. By our assumption the modules M and $N := K \oplus L$ are direct sum of virtually simple modules. Thus by Lemma 2.16, M and N are subisomorphic and so by Theorem 2.17, $M \cong N$; proving that ${}_R M$ is virtually semisimple.

$(2) \Rightarrow (3)$. By Theorem 2.14 and Proposition 3.3, every finitely generated left R-module is a direct sum of a semisimple module and a completely virtually semisimple projective module. Thus (3) is obtained by Proposition 3.1 and Theorem 2.17.

$(3) \Rightarrow (4)$. Since every simple R-module is either singular or projective, the condition (3) shows that every finitely generated left R-module is a direct sum of a singular and a projective module. Thus (4) is obtained by [14, Corollary 11.4].

$(4) \Rightarrow (1)$. Let M be a finitely generated left R-module. By assumption, M is extending with finite uniform dimension. Thus by [23, Lemma 6.43], M is a direct sum of uniform modules. So it is enough to show that every finitely generated uniform left R-module

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is virtually simple. Note that if U is a finitely generated uniform left R-module and $U \hookrightarrow \oplus_{i=1}^{k} V_i$ where each V_i is a virtually simple R-module, then by induction we can show that that $U \hookrightarrow V_j$ for some j. It follows that $U \cong V_j$ and the proof is complete. $\square$

Let R be a ring and M an R-module. An R-module N is *generated* by M or *M-generated* if there exists an epimorphism $M^{(\Lambda)} \longrightarrow N$ for some index set Λ. An R-module N is said to be *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. For an R-module M, we denote by $\sigma[M]$ the *full subcategory* of R-Mod whose objects are all R-modules subgenerated by M. It is clear that if $M = R$ then $\sigma[M]$ coincides with the category R-Mod.

Remark 3.5. As another application of the theory of virtually semisimple modules we shows that the term "cyclic" must be removed from statement (f) of [13, Proposition 13.3]. In fact, in Example 3.6 we show that the following statements are not equivalent.

(1) *Every module $N \in \sigma[M]$ is an extending module.*
(2) *Every cyclic module in $\sigma[M]$ is a direct sum of an M-projective module and a semisimple module.*

Example 3.6. Assume that ring D is the same as in Example 2.15 (example attributed to Cozzens). It is clear that $\sigma[D] = D$-Mod and since D is not left Artinian, so by [13, Proposition 13.5, Part g], there is a left D-module M such that M is not an extending module. On the other hand, D is a principal ideal V-domain and so by Corollary 2.9, every cyclic left D-module is a direct sum of a projective module and a semisimple module, and hence in the above (2) does not imply (1).

We note that the class of virtually simple modules is not closed under homomorphic image. For example the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ is not virtually semisimple but $\mathbb{Z}$ is clearly completely virtually semisimple $\mathbb{Z}$-module. Thus give the following definitions.

Definition 3.7. An R-module M is called *fully virtually semisimple* if for reach $N \leq M$, the R-module M/N is virtually semisimple. If $_RR$ (resp., R_R) is fully virtually semisimple, we then say that R is a *left* (resp., *right*) *fully virtually semisimple*. Also, a ring R is called a *fully virtually semisimple ring* if it is both a left and right fully virtually semisimple ring.

By Proposition 3.1 and the next proposition, we have the following irreversible implications for an R-module M:

M is fully virtually semisimple $\Rightarrow$ M is completely virtually semisimple $\Rightarrow$ M is a finite direct sum of virtually simple modules

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Proposition 3.8. *Every finitely generated fully virtually semisimple module M is completely virtually semisimple.*

Proof. Assume that $K \leq M$. It is well-known that there exists $L \leq M$ such that $L \oplus K \leq_e M$ and $K \cong (L \oplus K)/L \leq_e M/L$. Since M/L is finitely generated virtually semisimple, so $K \cong M/L$, by Lemma 2.1 (ii). Thus K is virtually semisimple. □

We conclude the paper with the following corollary that gives a partial solution to Question 1.9 raised in the introduction. In fact the following is an answer to the question in the case that "every left and every right cyclic R-module is virtually semisimple". However, finding the structure of non-commutative left fully virtually semisimple rings (rings each of whose left cyclic R-modules is virtually semisimple) is still an open question.

Corollary 3.9. *The following statements are equivalent for a ring R.*

(1) *Every left and every right cyclic R-module is virtually semisimple (i.e., R is a fully virtually semisimple ring).*

(2) *$R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where each D_i is a principal ideal V-domain.*

Proof. (1) $\Rightarrow$ (2). By Proposition 3.8, the ring R is a left and a right completely virtually semisimple ring. Thus by Theorem 2.7, $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$ where each D_i is a principal ideal domain. As seen in the proof (2) $\Rightarrow$ (3) of Theorem 2.14, we obtain each D_i is a left V-domain, and hence by Remark 2.8, each D_i is a principal ideal V-domain.

(2) $\Rightarrow$ (1) is by the second generalization Wedderburn-Artin Theorem. □

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