

REDUCTIONS TOWARDS A CHARACTERISTIC FREE PROOF OF THE CANONICAL ELEMENT THEOREM

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ABSTRACT. We reduce Hochster’s Canonical Element Conjecture (theorem since 2016) to a localization problem in a characteristic free way. We prove the validity of a new variant of the Canonical Element Theorem (CET) and explain how a characteristic free deduction of the new variant from the original CET would provide us with a characteristic free proof of the CET. We also show that the Balanced Big Cohen-Macaulay Module Theorem can be settled by a characteristic free proof if the big Cohen-Macaulayness of Hochster’s modification module can be deduced from the existence of a maximal Cohen-Macaulay complex in a characteristic free way.

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1. INTRODUCTION

Let (R, \mathfrak{m}_R) be a local ring of dimension n . The aim of this paper is to provide some results that we hope can be useful for obtaining a characteristic free proof of the following celebrated results in Commutative Algebra, all of which have been shown to be equivalent.

Direct Summand Theorem (DST): If R is module finite over a regular local subring P , then the inclusion $P \rightarrow R$ splits in the category of P -modules.

Monomial Theorem (MT): Let x_1, \dots, x_n be a system of parameters for R . Then

$$\forall t \geq 1 : x_1^t \cdots x_n^t \notin (x_1^{t+1}, \dots, x_n^{t+1}).$$

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Canonical Element Theorem (CE Property): Let F_\bullet be a minimal free resolution of R/\mathfrak{m}_R and let $K_\bullet(\mathbf{x}, R)$ be the Koszul complex of a system of parameters \mathbf{x} of R . Then $\phi_n \neq 0$ for any lifting $\phi_\bullet : K_\bullet(\mathbf{x}, R) \rightarrow F_\bullet$ of the natural epimorphism $R/(\mathbf{x}) \rightarrow R/\mathfrak{m}_R$.

Canonical Element Theorem (CET): Let F_\bullet be a free resolution of R/\mathfrak{m}_R and consider the natural epimorphism $\pi : F_n \rightarrow \text{syz}_n^R(R/\mathfrak{m}_R)$. Let η_R be the image of $[\pi]$ under the natural map

$$\phi : \text{Ext}_R^n(R/\mathfrak{m}_R, \text{syz}_n^R(R/\mathfrak{m}_R)) \rightarrow \varinjlim_{m \in \mathbb{N}} \text{Ext}_R^n(R/\mathfrak{m}_R^m, \text{syz}_n^R(R/\mathfrak{m}_R)) = H_{\mathfrak{m}_R}^n(\text{syz}_n^R(R/\mathfrak{m}_R)).$$

Then $\eta_R \neq 0$.

Improved New Intersection Theorem (INIT): Let $F_\bullet : 0 \rightarrow F_s \rightarrow F_{s-1} \rightarrow \cdots \rightarrow F_0 \rightarrow 0$ be a complex of finite free R -modules such that $\ell(H_i(F_\bullet)) < \infty$ for each $i \geq 1$ and $H_0(F_\bullet)$ has a minimal generator whose generated R -module has finite length. Then $s \geq n$.

The general form of the Direct Summand Theorem states that if R' is a (not necessarily local) Noetherian ring that is a module finite extension of a (not necessarily local) regular ring P , then $P \rightarrow R'$ splits (see [Ho83, Theorem (6.1)(1)]). However, this general case is reduced to our aforementioned local case: By arguments in [Ho83, last paragraph of page 539 and first paragraph of page 540] one can assume that P is a complete regular local ring and that the module finite extension R' of P is a domain. Such a module finite extension is necessarily local by [Co46, Theorem 7].

For more equivalent forms of the MT see [Du13], [Du16], [Du21], [Oh96], [AvIyNe18, Theorem 4.8], [KoLe97, Theorem (Rob₂)], Fact (P. Roberts), page 687], [SiSt03, Theorem 2], [StSt79] and [Ta22, Proposition 2.11].

In 1973, Hochster [Ho73] proposed the Direct Summand Conjecture and proved it in the case, where R contains a field. In 2002, Heitmann's breakthrough [He02] suggested that in the mixed characteristic case the conjecture could be settled using almost ring theory. In that case, the conjecture remained open until 2016 when Yves André [An18] proved it using perfectoid spaces. Thereafter, Bhargav Bhatt [Bh18] proved the derived variant of it, again using perfectoid spaces. Perfectoid algebras are mixed characteristic and non-Noetherian algebras. In the equicharacteristic case, proofs are known that do not use non-Noetherian rings. However, the known proofs still depend on the characteristic of R . In prime characteristic, one proof uses Grothendieck's non-Vanishing Theorem together with the Frobenius homomorphism. In equal-characteristic zero, the splitting is obtained using the normalized trace map.

For the following implications, proofs have been found that are characteristic free and do not use non-Noetherian rings.

$$\begin{array}{ccccc} \text{CET for } R & \xleftarrow{[\text{Ho83, Theorem (3.15)}]} & \text{CE Property for } R & \xleftarrow{[\text{Ho83, Theorem (2.6)}] \text{ and } [\text{Du87}]} & \text{INIT for } R \\ & & \Downarrow [\text{Ho83, Remarks (2.2)(7), page 506}] & & \\ & & \text{MT for } R & & \\ & & \Downarrow [\text{Ho73, Theorem 1}] & & \\ & & \text{DST for } R & & \end{array}$$

On the other hand, for deducing the CE Property from the DST for a residually prime characteristic local ring, the only known proof needs the assumption that every local ring with prime characteristic residue field satisfies the DST and the proof uses non-Noetherian rings (see [Ho83, Theorem (2.9)]).

In Section 5, we reduce the existence of a characteristic free proof of the CE Property (and hence of all other aforementioned conjectures) which avoids the use of non-Noetherian rings to the well-behavior of the CE Property under localization (see Theorem 5.1 and its sufficient condition (ii)). Moreover, in Section 4, we present a new variant of the Canonical Element Theorem (see Proposition-Definition 4.6) and we prove its validity (Corollary 4.19). Our proof of the validity of this new variant is based on the

existence of a balanced big Cohen-Macaulay module that behaves well under localization, and for this purpose we prove that the balanced big Cohen-Macaulay property for modules over excellent normal domains behaves well under localization (Theorem 4.17¹). Furthermore, a possible characteristic free deduction of the new variant of the Canonical Element Theorem from the original Canonical Element Theorem, if provable, can provide us with a characteristic free proof of the Canonical Element Theorem as shown in Theorem 5.1 (and its sufficient condition (i)).

As another main result of the paper, applying [IyMaScWa21, Proposition 4.3] and Theorem 5.1, we present a sufficient condition for obtaining a characteristic free proof for the existence of balanced big Cohen-Macaulay modules (Corollary 5.2).

We end the introduction by outlining the main steps of the proof of Theorem 5.1.

- (i) We reduce the CE Property to the case where R is a complete quasi-Gorenstein domain that is a locally complete intersection in codimension ≤ 1 by considering the same R -algebra as in the proof of [Ta17, Proposition 2.7].
- (ii) Suppose that R is a quasi-Gorenstein domain as in (i), and that S is the localization at the homogeneous maximal ideal of a generic linkage of R in the sense of Huneke and Ulrich [HuUl85]. Then S is an almost complete intersection domain that is a locally complete intersection in codimension ≤ 1 . Since S is an almost complete intersection domain with enough depth, we can apply Lemmas 2.12 and 2.20(ii) and an inductive argument to see that it satisfies the CE Property (cf. [DuGr08, step 2, page 238] for an analogous result with a different proof which seems to be characteristic dependent as it calls upon the proof of [Du94, Theorem 3.2]).
- (iii) Suppose that R is a quasi-Gorenstein domain as in (i). By the definition of generic linkage, there exists an excellent regular local ring A and linked ideals \mathfrak{a} and \mathfrak{b} of A such that $S := A/\mathfrak{a}$ is as in (ii) while A/\mathfrak{b} is a trivial deformation of R . We use the (factorial) extended Rees algebra $A[at, t^{-1}]$ ($A[at] \cong \text{Sym}_A(\mathfrak{a})$) of \mathfrak{a} as a bridge between S and A/\mathfrak{b} to deduce the CE Property for A/\mathfrak{b} (and thus for R) from the CE Property for S , under the validity of either of the two conditions of Theorem 5.1. The shape of this bridge can be observed via Proposition 3.6(iii) together with Lemma 3.2(ii). Here we make an essential use of Parts (iv) and (v) of Proposition 3.6 by which we were able to prove that the extended Rees algebra $A[at, t^{-1}]$ also satisfies the CE Property. The imposed conditions in Theorem 5.1 are only used for the deduction of the CE Property for the localization $A[at, t^{-1}]_{ht}$ of $A[at, t^{-1}]$ from the CE Property of $A[at, t^{-1}]$ (Step 7 of the proof of Theorem 5.1). In our proof of the CE Property, this deduction is thus the only obstacle to obtaining a characteristic free proof without any imposed condition.

In Part (iii) (above) we tried to follow the philosophy of Strooker and Stükrad [StSt79, page 158] who asked the following analogous question: If B and B' are linked local rings and B' satisfies the Small Cohen-Macaulay Conjecture, does B also satisfy the Small Cohen-Macaulay Conjecture? Lemma 3.2(ii) as well as the use of generic linkage were inspired by (the quasi-Gorenstein counterpart of) Ulrich [Ul84], as mentioned in our earlier paper [Ta17]. The generic linkage yields the linear type ideal \mathfrak{a} with factorial extended Rees Algebra $A[at, t^{-1}]$ (here [Hoc73] and [Hu81] are also used).

2. PRELIMINARIES

In this paper, rings are Noetherian and commutative with identity. Graded rings and graded modules appearing in this paper are \mathbb{Z} -graded and often graded rings are *graded local* in the sense that they admit a unique homogeneous ideal that is maximal with respect to all (homogeneous and non-homogeneous)

¹Alternatively, we could apply [BMPSTWW21, Proposition 2.11]. We refer to Remark 5.3 for some comments on the utility of Theorem 4.17 in parallel to [BMPSTWW21, Proposition 2.11].

proper ideals. For any $i \in \mathbb{Z}$ the i -component of a graded module M consisting of elements of degree i is denoted by $M_{[i]}$. If $(F_\bullet, \partial_\bullet^{F_\bullet})$ is a free resolution of a module M , then $\text{syz}_i^{F_\bullet}(M) := F_i/\text{im}(\partial_{i+1}^{F_\bullet})$ for any $i \geq 1$. Let $\mathbf{x} := x_1, \dots, x_t$ be a sequence of elements of a ring S and M be an S -module. The notation \mathbf{x}^v (respectively, $\mathbf{x}^{\underline{v}}$ where $\underline{v} = (v_1, \dots, v_t) \in \mathbb{N}^t$) denotes the sequence $x_1^{v_1}, \dots, x_t^{v_t}$ (respectively, $x_1^{v_1}, \dots, x_t^{v_t}$). For the definition of the Koszul complex $K_\bullet(\mathbf{x}; M)$ (and the Koszul homologies $H_i(\mathbf{x}; M)$) we follow [BrHe98, Definition 1.6.1] together with [BrHe98, The Koszul Complex of a Sequence, page 46].

2.1. Preliminaries on the Koszul complexes. The following lemmas concerning the Koszul complex are well-known, but we prove them for the convenience of the reader.

Lemma 2.1. *Let $\mathbf{x} := x_1, \dots, x_d$ be a sequence of elements of a ring S and*

$$(F_\bullet, \partial_\bullet^{F_\bullet}) := \cdots \rightarrow F_n \xrightarrow{\partial_n^{F_\bullet}} \cdots \rightarrow F_{n-1} \xrightarrow{\partial_{n-1}^{F_\bullet}} \cdots \rightarrow F_1 \xrightarrow{\partial_1^{F_\bullet}} F_0 \rightarrow 0$$

be an acyclic complex. Suppose that $\varphi_\bullet, \varphi'_\bullet : K_\bullet(\mathbf{x}; S) \rightarrow F_\bullet$ are chain maps of complexes such that $\overline{\varphi_0}, \overline{\varphi'_0} : S/\mathbf{x}S \rightarrow H_0(F_\bullet)$ coincide.

(i) $\varphi_0 - \varphi'_0$ is null-homotopic.

(ii) $\varphi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) - \varphi'_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) \in (x_1, \dots, x_d)(F_d/\text{im}(\partial_{d+1}^{F_\bullet}))$.

Proof. (i) Let $\pi : F_0 \rightarrow H_0(F_\bullet)$ be the natural epimorphism. By our hypothesis, $\pi(\varphi_0 - \varphi'_0) = 0$. As $K_0(\mathbf{x}; S)$ is free and

$$F_1 \xrightarrow{\partial_1^{F_\bullet}} F_0 \xrightarrow{\pi} H_0(F_\bullet)$$

is exact, there exists $g_0 : K_0(\mathbf{x}; S) \rightarrow F_1$ such that $\partial_1^{F_\bullet} \circ g_0 = \varphi_0 - \varphi'_0$. Set, $g_{-1} = 0$ as well. Assume inductively that for some $i \geq 0$, $g_i : K_i(\mathbf{x}; S) \rightarrow F_{i+1}$ and $g_{i-1} : K_{i-1}(\mathbf{x}; S) \rightarrow F_i$ are constructed such that

$$(2.1) \quad \varphi_i - \varphi'_i = \partial_{i+1}^{F_\bullet} \circ g_i + g_{i-1} \circ \partial_i^{K_\bullet(\mathbf{x}; S)}.$$

It suffices to find $g_{i+1} : K_{i+1}(\mathbf{x}; S) \rightarrow F_{i+2}$ with the desired property. But,

$$\begin{aligned} \partial_{i+1}^{F_\bullet} \circ ((\varphi_{i+1} - \varphi'_{i+1}) - (g_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)})) &= \partial_{i+1}^{F_\bullet} \circ (\varphi_{i+1} - \varphi'_{i+1}) - \partial_{i+1}^{F_\bullet} \circ (g_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)}) \\ &= \partial_{i+1}^{F_\bullet} \circ (\varphi_{i+1} - \varphi'_{i+1}) - (\partial_{i+1}^{F_\bullet} \circ g_i) \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} \\ &\stackrel{\text{Eq. (2.1)}}{=} \partial_{i+1}^{F_\bullet} \circ (\varphi_{i+1} - \varphi'_{i+1}) - (\varphi_i - \varphi'_i - g_{i-1} \circ \partial_i^{K_\bullet(\mathbf{x}; S)}) \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} \\ &= \partial_{i+1}^{F_\bullet} \circ (\varphi_{i+1} - \varphi'_{i+1}) - (\varphi_i - \varphi'_i) \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} \\ &= \varphi_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} - \varphi'_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} - (\varphi_i - \varphi'_i) \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)} \\ &= 0 \end{aligned}$$

Thus we get the map

$$(\varphi_{i+1} - \varphi'_{i+1}) - (g_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)}) : K_{i+1}(\mathbf{x}; S) \rightarrow \ker(\partial_{i+1}^{F_\bullet}).$$

Thence, since $F_{i+2} \xrightarrow{\partial_{i+2}^{F_\bullet}} F_{i+1} \xrightarrow{\partial_{i+1}^{F_\bullet}} F_i$ is exact, there exists some $g_{i+1} : K_{i+1}(\mathbf{x}; S) \rightarrow F_{i+2}$ such that $\partial_{i+2}^{F_\bullet} \circ g_{i+1} = (\varphi_{i+1} - \varphi'_{i+1}) - (g_i \circ \partial_{i+1}^{K_\bullet(\mathbf{x}; S)})$ as required.

(ii) Since φ_\bullet and φ'_\bullet are null-homotopic by the previous part, following the notation of the previous part, we have $\varphi_d(1) - \varphi'_d(1) = \partial_{d+1}^{F_\bullet} \circ g_d(1) + g_{d-1} \circ \partial_d^{K_\bullet(\mathbf{x}; S)}(1)$. So modulo $\text{im}(\partial_{d+1}^{F_\bullet})$ the image of $\varphi_d(1) - \varphi'_d(1)$ coincides with the image of

$$g_{d-1} \circ \partial_d^{K_\bullet(\mathbf{x}; S)}(1) = g_{d-1}(x_1, -x_2, \dots, (-1)^{d+1}x_d) \in (x_1, \dots, x_d)F_d. \quad \square$$

Lemma 2.2. *Let S be a ring and let $\mathbf{x} := x_1, \dots, x_d$ and $\mathbf{y} := y_1, \dots, y_d$ be sequences of elements of S such that $(\mathbf{y}) \subseteq (\mathbf{x})$.*

- (i) *There exists a chain map $\varphi_\bullet : K_\bullet(\mathbf{y}; S) \rightarrow K_\bullet(\mathbf{x}; S)$ of Koszul complexes such that $\varphi_0 = id_S$. In particular, φ_\bullet lifts the natural epimorphism $S/(\mathbf{y}) \rightarrow S/(\mathbf{x})$.*
- (ii) *Suppose that $\mathbf{y} = \mathbf{x}^{\underline{v}}$ for some $\underline{v} = (v_1, \dots, v_d) \in \mathbb{N}^d$. Then a chain map $\varphi_\bullet : K_\bullet(\mathbf{x}^{\underline{v}}; S) \rightarrow K_\bullet(\mathbf{x}; S)$ exists such that $\varphi_0 = id_S$, and moreover $\varphi_d : S \rightarrow S$ is the multiplication map by $x_1^{v_1-1} \dots x_d^{v_d-1}$ ($S = K_d(\mathbf{x}^{\underline{v}}; S) = K_d(\mathbf{x}; S)$).*

Proof. (i) For each $1 \leq i \leq d$, by our hypothesis there are elements $s_{1,i}, \dots, s_{d,i}$ such that $y_i = \sum_{j=1}^d s_{ji}x_j$.

Let $\mathbf{A} := \begin{pmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,d} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,d} \\ \vdots & \vdots & \vdots & \vdots \\ s_{d,1} & s_{d,2} & \cdots & s_{d,d} \end{pmatrix}$, $\mathbf{B} := (x_1 \ x_2 \ \cdots \ x_d)$ and $\mathbf{C} := (y_1 \ y_2 \ \cdots \ y_d)$ be matrices

whose corresponding homomorphisms of free modules, $f_{\mathbf{A}}$, $f_{\mathbf{B}}$ and $f_{\mathbf{C}}$ yield the commutative diagram

$$\begin{array}{ccc} S^d & \xrightarrow{f_{\mathbf{C}}} & S \\ \downarrow f_{\mathbf{A}} & & \downarrow id_S \\ S^d & \xrightarrow{f_{\mathbf{B}}} & S \end{array}$$

Consequently, according to the definition of the Koszul complex as well as [BrHe98, Proposition 1.6.8 and its preceding paragraph] $f_{\mathbf{A}}$ and id_S in the above commuting square extend to a chain map of Koszul complexes $\bigwedge f_{\mathbf{A}} : K_\bullet(\mathbf{y}; S) \rightarrow K_\bullet(\mathbf{x}; S)$ that is a homomorphism of S -DG algebras (see also [BrHe98, page 41], where $\bigwedge \varphi$ is defined). Note that since $\bigwedge f_{\mathbf{A}}$ is an S -algebra homomorphism so $(\bigwedge f_{\mathbf{A}})_0 = id_S$. Hence the desired conclusion follows by setting $\varphi_\bullet := \bigwedge f_{\mathbf{A}}$.

(ii) In this case, we can set the matrix \mathbf{A} to be the diagonal matrix $\text{diag}(x_1^{v_1-1}, \dots, x_d^{v_d-1})$. Hence, following the notation used in the proof of part (i) we have

$$\begin{aligned} \varphi_d(1_{K_d(\mathbf{x}^{\underline{v}}; S)}) &= (\bigwedge f_{\mathbf{A}})_d(e_1 \wedge \cdots \wedge e_d) \\ &= f_{\mathbf{A}}(e_1) \wedge \cdots \wedge f_{\mathbf{A}}(e_d) && \text{(see [BrHe98, page 41], where } \bigwedge \varphi \text{ is defined)} \\ &= (x_1^{v_1-1} \cdots x_d^{v_d-1})(e_1 \wedge \cdots \wedge e_d) \\ &= (x_1^{v_1-1} \cdots x_d^{v_d-1})1_{K_d(\mathbf{x}; S)}. \end{aligned}$$

□

2.2. Preliminaries on quasi-Gorenstein and almost complete intersection rings, etc. For the definition and some properties of the canonical module ω_S of a local ring S see [BrSh13, Chapter 12] (see also [Ao83]).

Remark 2.3. Let S be a local ring which is a homomorphic image of a Gorenstein local ring G . Suppose that $S = G/\mathfrak{g}$ and that $l = \text{height}(\mathfrak{g})$. By [BrSh13, Remarks 12.1.3(iii)] and [BrHe98, Corollary 2.1.4], $\omega_S \cong \text{Ext}_G^l(S, G)$. In particular, if \mathfrak{c} is a complete intersection contained in \mathfrak{g} with the same height as \mathfrak{g} , then

$$(2.2) \quad \omega_S \cong \text{Ext}_G^l(S, G) \stackrel{[\text{Ma89, Lemma 2(i), page 140}]}{\cong} \text{Hom}_G(G/\mathfrak{g}, G/\mathfrak{c}) \cong (\mathfrak{c} : \mathfrak{g})/\mathfrak{c}.$$

Definition 2.4. An ideal \mathfrak{a} of a regular local ring A is called an *almost complete intersection ideal* if $\mu(\mathfrak{a}) = \text{height}(\mathfrak{a}) + 1$, where μ denotes the minimal number of generators. An *almost complete intersection ring* is a residue ring of a regular local ring by an almost complete intersection ideal.

Note that if \mathfrak{a} is an almost complete intersection ideal of a regular local ring A , then by [BrHe98, Exercise 1.2.21] there exists a complete intersection ideal \mathfrak{c} of A with the same height as \mathfrak{a} such that $\mathfrak{a} = \mathfrak{c} + (h)$ for some $h \in A$.

Definition 2.5. A local ring S is called *quasi-Gorenstein* if it admits a canonical module that is a free module, necessarily of rank 1. An ideal \mathfrak{b} of a regular local ring A is called a *quasi-Gorenstein ideal* if A/\mathfrak{b} is a quasi-Gorenstein ring.

Definition 2.6. Let \mathfrak{a} be an ideal of a ring A . Then \mathfrak{a} is called *unmixed* if $\dim(A/\mathfrak{p}) = \dim(A/\mathfrak{a})$ for all $\mathfrak{p} \in \text{ass}(\mathfrak{a})$.

Definition 2.7. Let A be a regular (graded) local ring and $\mathfrak{a}, \mathfrak{b}$ be (homogeneous) ideals of A . Then we say that \mathfrak{a} and \mathfrak{b} are *linked* over a (homogeneous) complete intersection ideal $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$ provided $\mathfrak{b} = \mathfrak{c} : \mathfrak{a}$, $\mathfrak{a} = \mathfrak{c} : \mathfrak{b}$. In this case, we say that A/\mathfrak{a} and A/\mathfrak{b} are *linked rings*.

Remark 2.8. From the above definition of linkage, it is easily concluded that $\text{height}(\mathfrak{a}) = \text{height}(\mathfrak{c}) = \text{height}(\mathfrak{b})$ provided the ideals \mathfrak{a} and \mathfrak{b} are linked over the complete intersection \mathfrak{c} and $\mathfrak{a}, \mathfrak{b}$ are proper (because any complete intersection ideal is unmixed). Moreover, linked ideals are necessarily unmixed (see [Ly12, Lemma 2.1]).

Some of our results in this paper exploit the linkage of unmixed almost complete intersection ideals and quasi-Gorenstein ideals. Although the next lemma is well-known to experts, we include it in our paper for the convenience of the reader.

Lemma 2.9. Let A be a regular local ring.

- (i) Let \mathfrak{b} be a quasi-Gorenstein ideal of A of height g and \mathfrak{c} be a height g complete intersection ideal contained properly in \mathfrak{b} . Then $\mathfrak{a} := \mathfrak{c} : \mathfrak{b}$ is an unmixed almost complete intersection ideal $\mathfrak{a} = \mathfrak{c} + (h) \subset A$. Moreover, \mathfrak{a} and \mathfrak{b} are linked over \mathfrak{c} .
- (ii) Let $\mathfrak{a} = \mathfrak{c} + (h)$ be a height g unmixed almost complete intersection ideal of A containing the complete intersection ideal \mathfrak{c} of height g . Then $\mathfrak{b} := \mathfrak{c} : \mathfrak{a}$ is a quasi-Gorenstein ideal. Moreover, \mathfrak{a} and \mathfrak{b} are linked over \mathfrak{c} .

Proof.

- (i) Since the canonical module is always S_2 ([Ao83, (1.10)]) and $A/\mathfrak{b} \cong \omega_{A/\mathfrak{b}}$, so A/\mathfrak{b} satisfies the (S_2) -condition. Immediately \mathfrak{b} is an unmixed ideal since A/\mathfrak{b} is (S_1) and equidimensional. Then \mathfrak{a} and \mathfrak{b} are linked by [Sc82, Proposition 2.2] (the term ‘‘pure height’’ in the statement of [Sc82, Proposition 2.2] points to the same concept of unmixedness as in our paper) and so \mathfrak{a} is unmixed by Remark 2.8. Also in view of the display (2.2), $\mathfrak{a}/\mathfrak{c} \cong \omega_{A/\mathfrak{b}} \cong A/\mathfrak{b}$ and so $\mathfrak{a}/\mathfrak{c}$ is a cyclic module. Hence $\mathfrak{a} = \mathfrak{c} + (h)$ (generated minimally, so $\mu(\mathfrak{a}) = g + 1$) for some $h \in A$. Note that $\mathfrak{c} + (h)$ is not a complete intersection, since otherwise $\mathfrak{b} = \mathfrak{c} : \mathfrak{a} = \mathfrak{c}$ violates our hypothesis that $\mathfrak{c} \subsetneq \mathfrak{b}$. Consequently, \mathfrak{a} is an almost complete intersection.
- (ii) Again by [Sc82, Proposition 2.2], \mathfrak{a} and \mathfrak{b} are linked. Thus, in view of the display (2.2), $\omega_{A/\mathfrak{b}} \cong \mathfrak{a}/\mathfrak{c}$. Moreover, as $(\mathfrak{c} : h) = \mathfrak{b}$, $\mathfrak{a}/\mathfrak{c} \cong A/\mathfrak{b}$ and A/\mathfrak{b} is quasi-Gorenstein. \square

Definition 2.10. Let H be an Artinian S -module. A prime ideal \mathfrak{p} of S is called an *attached prime ideal* of A if $\mathfrak{p} = 0 :_S (H/N)$ for some submodule N of H . The set of attached prime ideals of H is denoted by $\text{Att}(H)$.

Lemma 2.11. Let H be an Artinian S -module.

- (i) $\text{Att}(H)$ is a finite set.
- (ii) If M is a finitely generated S -module and (S, \mathfrak{m}_S) is local, then

$$\text{Att}\left(\text{Hom}_S(M, E(S/\mathfrak{m}_S))\right) = \text{Ass}(M).$$

(iii) Let $x \in S$. Then $x \notin \bigcup_{\mathfrak{p} \in \text{Att}(H)} \mathfrak{p}$ if and only if the multiplicative map by x on H is surjective.

Proof. The statements follow from [BrSh13, Chapter 7, Corollary 10.2.20, Proposition 7.2.11(i), respectively]. \square

Lemma 2.12. Let (S, \mathfrak{m}_S) be a local ring of dimension ≥ 2 admitting a canonical module ω_S . If

$$\mathfrak{m}_S \in \text{Att}(H_{\mathfrak{m}_S}^{\dim(S)-1}(S))$$

then $\text{depth}(\omega_S) = 2$.

Proof. This follows from [AoGo85, Lemma 2.1(2)(i)]. \square

Remark 2.13. Under the conditions that S is (S_2) with Cohen-Macaulay formal fibers and \widehat{S} is equidimensional, the above lemma and its converse can be proved via some results of [Di05] and [DiJa11]. For the details see Remark 2.7 of the arXiv version of [TaTu18]. For another proof of this lemma and its converse assuming only that $\text{depth}(S) > 0$ and $\dim(S) \geq 3$, see Lemma 2.5 of the arXiv version of [TaTu18].

The following lemma will be used in the proof of Theorem 5.1.

Lemma 2.14. Let (S, \mathfrak{m}_S) be a local ring and X be an indeterminate over S . Then, for all i the multiplication map by X on $H_{(\mathfrak{m}_S, X)}^i(S[X]_{(\mathfrak{m}_S, X)})$ is surjective.

Proof. $H_{(\mathfrak{m}_S, X)}^0(S[X]_{(\mathfrak{m}_S, X)}) = 0$ in view of the regularity of the element X . Fix some $1 \leq i \leq \dim(S) + 1$. By [BrSh13, 4.3.2 Flat Base Change Theorem], $H_{(\mathfrak{m}_S)}^i(S[X]) \cong H_{\mathfrak{m}_S}^i(S)[X]$, so $\Gamma_{(X)}(H_{(\mathfrak{m}_S)}^i(S[X])) = 0$. Consequently, from [Sc98, Corollary 1.4] we get

$$(2.3) \quad H_{(\mathfrak{m}_S, X)}^i(S[X]) \cong H_{(X)}^1(H_{(\mathfrak{m}_S)}^{i-1}(S[X])).$$

But X acts surjectively on $H_{(\mathfrak{m}_S)}^{i-1}(S[X])_X$, while $H_{(X)}^1(H_{(\mathfrak{m}_S)}^{i-1}(S[X]))$ is a quotient of $H_{(\mathfrak{m}_S)}^{i-1}(S[X])_X$ by [BrSh13, Corollary 2.2.21]. From this fact and the display (2.3) we conclude that X acts surjectively on $H_{(\mathfrak{m}_S, X)}^i(S[X]) \cong H_{(\mathfrak{m}_S, X)}^i(S[X]_{(\mathfrak{m}_S, X)})$ (since (\mathfrak{m}_S, X) is a maximal ideal, so the localization map at (\mathfrak{m}_S, X) is an isomorphism on each (\mathfrak{m}_S, X) -torsion module). \square

2.3. Preliminaries on the CE Property. In this subsection, we need to use the concept of projective resolution of bounded below complexes. In the sequel, \simeq denotes the quasi-isomorphism of complexes.

Definitions 2.15. Let S be a (Noetherian commutative) ring. A complex C_\bullet of S -modules is called a *bounded below complex* if $C_i = 0$ for $i \ll 0$. It is called a *homologically bounded below complex* if $H_i(C) = 0$ for all $i \ll 0$. It is called a *degreewise finite complex* provided C_i is a finitely generated S -module for each i . A *projective resolution* of a homologically bounded below complex C_\bullet is a bounded below complex P_\bullet consisting of projective S -modules together with a quasi-isomorphism $\pi_\bullet : P_\bullet \xrightarrow{\sim} C_\bullet$ (see [Ch00, Definitions (A.3.1)(P)]).

Remark 2.16. (see [AvFo91, Remarks 1.7]) Let S be a (Noetherian commutative) ring and C_\bullet be a homologically bounded below complex of S -modules with finitely generated homologies. Then C_\bullet has a projective resolution $\pi_\bullet : P_\bullet \xrightarrow{\sim} C_\bullet$ such that each P_i is a finitely generated free module and such that $P_i = 0$ for each $i < \inf\{j \in \mathbb{Z} : H_j(C_\bullet) \neq 0\}$ ([AvFo91, Remarks 1.7]).

The following lemma will be used in the proof of Lemma 2.18. This lemma is pointed out to the author by the referee in order to obviate the need for spectral sequences in the (previous) author's proof.

Lemma 2.17. *If L_\bullet, C_\bullet are both bounded below complexes of S -modules (or both bounded above) and the complex $L_\bullet \otimes_S C_\bullet$ is exact at $L_{i-m} \otimes_S C_m$ for all m , then $H_i(L_\bullet \otimes_S C_\bullet) = 0$.*

Proof. We only prove the bounded below case. By a naive shifting if necessary, to simplify the notation and without loss of generality, we may and we do assume that $C_j = 0, L_j = 0$ provided $j < 0$. So at the i -th spot the complex $L_\bullet \otimes_S C_\bullet$ is

$$\bigoplus_{j=0}^{i+1} L_{i-j+1} \otimes_S C_j \xrightarrow{\partial_{i+1}^{L_\bullet \otimes_S C_\bullet}} \bigoplus_{j=0}^i L_{i-j} \otimes_S C_j \xrightarrow{\partial_i^{L_\bullet \otimes_S C_\bullet}} \bigoplus_{j=0}^{i-1} L_{i-j-1} \otimes_S C_j$$

with well-known differentials. Let $h \in H_i(L_\bullet \otimes_S C_\bullet)$. Assume, inductively, that there is a representative $(\alpha_j)_{j=0}^i \in \ker \partial_i^{L_\bullet \otimes_S C_\bullet}$, where $\alpha_j \in L_{i-j} \otimes_S C_j$, such that $h = [(\alpha_j)_{j=0}^i]$ and $\alpha_j = 0$ for each $j > i$. We claim that then there is a representative of h as

$$(2.4) \quad h = [(\alpha'_j)_{j=0}^i] \text{ s.t. } \alpha'_j \in L_{i-j} \otimes_S C_j, \quad \alpha'_i = \alpha'_{i+1} = \dots = \alpha'_i = 0.$$

Then the statement will follow from our claim and the induction.

To prove the claim, we notice that from $0 = \alpha_{i+1} \in L_{i-l+1} \otimes_S C_{l+1}$ and $(\alpha_j)_{j=0}^i \in \ker \partial_i^{L_\bullet \otimes_S C_\bullet}$ we get

$$\partial_{i-l}^{L_\bullet \otimes_S C_\bullet}(\alpha_l) = \pm \partial_{i-l+1}^{L_{i-l-1} \otimes_S C_\bullet}(\alpha_{l+1}) = 0.$$

From this and the exactness of $L_\bullet \otimes_S C_\bullet$ at $L_{i-l} \otimes_S C_l$ (by our assumption), we get some $\gamma \in L_{i-l+1} \otimes_S C_l$ such that $\partial_{i-l+1}^{L_\bullet \otimes_S C_\bullet}(\gamma) = \alpha_l$. Therefore

$$\begin{aligned} h &= [(\alpha_j)_{j=0}^i] = [(\alpha_j)_{j=0}^i] + \partial_{i+1}^{L_\bullet \otimes_S C_\bullet}(-\gamma) = [(\alpha_j)_{j=0}^i] - [(0, \dots, 0, \pm \partial_i^{L_{i-l+1} \otimes_S C_\bullet}(\gamma), \alpha_l, 0, \dots, 0)] \\ &= [(\alpha_0, \dots, \alpha_{l-2}, \alpha_{l-1} \mp \partial_l^{L_{i-l+1} \otimes_S C_\bullet}(\gamma), 0, \dots, 0,)] \end{aligned}$$

can be represented as claimed in display (2.4), as was to be proved. \square

The next lemma is a new result which will be used in the proof of Lemma 2.20(i). Moreover, the following lemma can be used to provide an alternative proof for [DMTT19, Corollary 2.8(i)] avoiding the use of big Cohen-Macaulay modules; even more, using the next lemma one can deduce that a non-exact complex $G_\bullet := 0 \rightarrow G_s \rightarrow \dots \rightarrow G_0 \rightarrow 0$ of (S, \mathfrak{m}_S) -modules as in [DMTT19, Corollary 2.8(i)] satisfies the analogue of the Improved New Intersection Theorem (which is stronger than the New Intersection Theorem).

Lemma 2.18. *Let S be a (Noetherian commutative) ring and*

$$C_\bullet := 0 \rightarrow C_s \rightarrow C_{s-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

be a complex of S -modules with finitely generated homologies such that $H_0(C_\bullet) \neq 0$. Suppose that $\text{pd}_S(C_i) = \ell$ for each $0 \leq i \leq s$, where $\ell \geq 1$. Then there is a projective resolution $\pi_\bullet : P_\bullet \xrightarrow{\sim} C_\bullet$ of C_\bullet where

$$P_\bullet = 0 \rightarrow P_t \rightarrow P_{t-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

with $t \leq s + \ell$ (i.e. P_\bullet has length $\leq s + \ell$) and such that P_\bullet consists of finitely generated projective modules.

Proof. By Remark 2.16, there is a projective resolution $\pi'_\bullet : P'_\bullet \xrightarrow{\sim} C_\bullet$ of C_\bullet consisting of finitely generated projective S -modules such that $P'_i = 0$ for each $i \leq 0$.

If P'_\bullet already has length $\leq s + \ell$, then we are done by setting $P_\bullet := P'_\bullet$ and $\pi_\bullet := \pi'_\bullet$. So we assume that P'_\bullet is either an infinite complex or it has length $> s + \ell$. As $P'_\bullet \simeq C_\bullet$ and C_\bullet has length s so

$$(2.5) \quad \forall i \geq s + 1, \quad H_i(P'_\bullet) = 0.$$

Let M be an arbitrary S -module. We aim to show that

$$(2.6) \quad \forall i \geq s + \ell + 1, \quad H_i(M \otimes_S P'_\bullet) = 0.$$

Let L_\bullet^M be an S -free resolution of M . Then by [Ch00, (A.4.1) Preservation of Quasi-Isomorphisms and Equivalences],

$$M \otimes_S P'_\bullet \xrightarrow{(- \otimes_S P'_\bullet) \text{ preserves quasi-iso.}} \simeq L_\bullet^M \otimes_S P'_\bullet \xrightarrow{(L_\bullet^M \otimes_S -) \text{ preserves quasi-iso.}} \simeq L_\bullet^M \otimes_S C_\bullet.$$

Hence display (2.6) follows if we can show that

$$(2.7) \quad \forall i \geq s + \ell + 1, \quad H_i(L_\bullet^M \otimes_S C_\bullet) = 0.$$

If $q, p \in \mathbb{N}_0$ are such that $i = q + p \geq s + \ell + 1$, then either $p \geq s + 1$ or $q \geq \ell + 1$, so

$$(2.8) \quad H_q(L_\bullet^M \otimes_S C_p) \cong \text{Tor}_q^S(M, C_p) = 0, \quad (\text{either } C_p = 0 \text{ (if } p \geq s + 1) \text{ or } \text{pd}_S(C_p) = \ell < q).$$

Now if we apply Lemma 2.17 to the vanishings in display (2.8) for each $i = p + q \geq s + \ell + 1$, then the desired vanishings in display (2.7) and thence in (2.6) immediately follow.

As $H_i(P'_\bullet) = 0$ for each $i \geq s + 1$ by display (2.5), so in the hard truncation

$$P'_{\geq s+\ell} := \cdots \rightarrow P'_{s+\ell+3} \rightarrow P'_{s+\ell+2} \rightarrow P'_{s+\ell+1} \rightarrow P'_{s+\ell} \rightarrow 0,$$

of P'_\bullet , all homologies, but $H_{s+\ell}(P'_{\geq s+\ell}) = P'_{s+\ell}/\text{im}(\partial_{s+\ell+1}^{P'_\bullet})$, are zero. This together with display (2.6) implies that

$$\text{Tor}_1^S(M, P'_{s+\ell}/\text{im}(\partial_{s+\ell+1}^{P'_\bullet})) = H_{s+\ell+1}(M \otimes_S P'_\bullet) = 0$$

for any S -module M . Consequently, $P'_{s+\ell}/\text{im}(\partial_{s+\ell+1}^{P'_\bullet})$ is a projective module as finitely generated flat modules are projective. Let P_\bullet be the soft truncation

$$0 \rightarrow P'_{s+\ell}/\text{im}(\partial_{s+\ell+1}^{P'_\bullet}) \xrightarrow{\overline{\partial_{s+\ell}^{P'_\bullet}}} P'_{s+\ell-1} \xrightarrow{\partial_{s+\ell-1}^{P'_\bullet}} P'_{s+\ell-2} \rightarrow \cdots \rightarrow P'_0 \rightarrow 0$$

of P'_\bullet , which is a complex of finitely generated projective modules of length $\leq s + \ell$. To define $\pi_\bullet : P_\bullet \rightarrow C_\bullet$, set $\pi_i = \pi'_i$ for each $0 \leq i < s + \ell$ and set $\pi_{s+\ell} : P'_{s+\ell}/\text{im}(\partial_{s+\ell+1}^{P'_\bullet}) \rightarrow C_{s+\ell}(= 0)$ to be the zero homomorphism. Then $\pi_\bullet : P_\bullet \rightarrow C_\bullet$ is a chain map and it is a quasi-isomorphism. \square

Remark 2.19. We bring to the reader's attention that in the statement of Lemma 2.18 the complex C_\bullet is not assumed to be a degreewise finite complex and only the finiteness of its homologies is required.

The first part of the following lemma will be used in the last step of the proof of Theorem 5.1. Namely, therein we arrive at a possibly non-local ring Q with a maximal ideal \mathfrak{m} such that $(Q/(\mathbf{x}), (\mathfrak{m}))$ is a local ring for some regular sequence $\mathbf{x} \subseteq \mathfrak{m}$. Then Lemma 2.20(i) deduces the CE Property for $Q/(\mathbf{x})$ from the validity of a similar CE Property for Q . Lemma 2.20(ii) shows that the CE Property is preserved under deformation, but under a further condition that is (fortunately) available in the situation of the proof of Theorem 5.1.

Lemma 2.20. *The following statements hold.*

- (i) *Let Q be a Noetherian ring and \mathfrak{m} be a maximal ideal of height d . Let $\mathbf{x} \subseteq \mathfrak{m}$ be a regular sequence of Q such that $(Q/\mathbf{x}Q, (\mathfrak{m}))$ is a local ring. Let F_\bullet be a free resolution of Q/\mathfrak{m} . Presume that Q satisfies the following CE Property:*

For some sequence $\mathbf{z} := z_1, \dots, z_d$ ($d = \text{height}(\mathfrak{m})$) with $\sqrt{(\mathbf{z})} = \mathfrak{m}$ and for some chain map $\phi_\bullet : K_\bullet(\mathbf{z}; Q) \rightarrow F_\bullet$ lifting the natural epimorphism $Q/(\mathbf{z}) \rightarrow Q/\mathfrak{m}$, it is the case that

$$\forall v \in \mathbb{N}, \quad z_1^{v-1} \cdots z_d^{v-1} \phi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) \notin (z_1^v, \dots, z_d^v) \text{syz}_d^{F_\bullet}(Q/\mathfrak{m}).$$

Then the local ring $Q/\mathbf{x}Q$ satisfies the CE Property.

(ii) Let (S, \mathfrak{m}_S) be a local ring that is a homomorphic image of a Gorenstein local ring. Suppose that x is a regular element of S such that S/xS satisfies the CE Property. If

$$x \notin \bigcup_{\mathfrak{p} \in \text{Att}(H_{\mathfrak{m}_S}^{\dim(S)-1}(S))} \mathfrak{p},$$

then S also satisfies the CE Property.

Proof. (i) By virtue of [Du87], it suffices to show that $Q/\mathbf{x}Q$ satisfies the Improved New Intersection Conjecture. So let

$$G_\bullet := 0 \rightarrow G_s \rightarrow G_{s-1} \rightarrow \cdots \rightarrow G_0 \rightarrow 0$$

be a complex of finite free $(Q/\mathbf{x}Q)$ -modules such that $\ell(H_i(G_\bullet)) < \infty$ for $i > 0$ and $H_0(G_\bullet)$ admits a minimal generator h of finite length. We should show that $\dim(Q/\mathbf{x}Q) \leq s$ (we assume that $G_s \neq 0$).

Let ℓ be the length of the regular sequence \mathbf{x} . So $\ell = \text{pd}_Q(G_j)$ for any $0 \leq j \leq s$, as $G_j = (Q/\mathbf{x}Q)^{n_j}$ for some $n_j \in \mathbb{N}_0$. Thus applying Lemma 2.18, there exists a complex

$$P_\bullet := 0 \rightarrow P_t \rightarrow P_{t-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

of finitely generated projective Q -modules with $t \leq \ell + s$ such that it is quasi-isomorphic to G_\bullet . Consequently $\mathfrak{m}^u H_i(P_\bullet) = 0$ for all $i > 0$, and $\mathfrak{m}^u h = 0$ for some $u \in \mathbb{N}$. Therefore, considering the sequence \mathbf{z} as in the statement of the lemma, we have $(\mathbf{z})^u H_i(P_\bullet) = (\mathbf{z})^u h = 0$ for some $u \in \mathbb{N}$. Thus, arguing as in the proof of [Ho83, Theorem (2.6)] we can construct a chain map of complexes

$$g_\bullet : K_\bullet(\mathbf{z}^v; Q) \rightarrow P_\bullet$$

for some $v \geq u$ such that $\overline{g}_0 : Q/\mathbf{z}^v Q \rightarrow H_0(P_\bullet)$ is given by $1 + (\mathbf{z}^v) \mapsto h$.

Since $h \in H_0(P_\bullet) \cong H_0(G_\bullet)$ is a minimal generator so there is an epimorphism

$$\pi : H_0(P_\bullet) \twoheadrightarrow H_0(P_\bullet)/\mathfrak{m}H_0(P_\bullet) \xrightarrow{\text{projection}} Q/\mathfrak{m}, \quad \pi(h) = 1 + \mathfrak{m}.$$

As P_\bullet is a complex of projective Q -modules and $F_\bullet \rightarrow Q/\mathfrak{m} \rightarrow 0$ is an exact complex (F_\bullet is the free resolution mentioned in the statement) so there is a chain map of complexes

$$g'_\bullet : P_\bullet \rightarrow F_\bullet, \quad \overline{g'_0} = \pi$$

where $\overline{g'_0} : H_0(P_\bullet) \rightarrow H_0(F_\bullet) = Q/\mathfrak{m}$ is the induced map by g'_0 . Consequently,

$$\psi_\bullet := g'_\bullet \circ g_\bullet : K_\bullet(\mathbf{z}^v; Q) \rightarrow F_\bullet$$

is a chain map lifting the natural surjection $Q/\mathbf{z}^v Q \rightarrow Q/\mathfrak{m}$. Also, we have another chain map

$${}_v\phi_\bullet := \phi_\bullet \circ {}_v\lambda_\bullet : K_\bullet(\mathbf{z}^v; Q) \rightarrow F_\bullet$$

that is lifting the natural surjection $Q/\mathbf{z}^v Q \rightarrow Q/\mathfrak{m}$, where $\phi_\bullet : K_\bullet(\mathbf{z}; Q) \rightarrow F_\bullet$ is the chain map mentioned in the statement, and ${}_v\lambda_\bullet : K_\bullet(\mathbf{z}^v; Q) \rightarrow K_\bullet(\mathbf{z}; Q)$ is as in Lemma 2.2. Note that

$${}_v\phi_d(1) = \phi_d({}_v\lambda_d(1)) \stackrel{\text{Lemma 2.2(ii)}}{=} \phi_d(z_1^{v-1} \cdots z_d^{v-1}) = z_1^{v-1} \cdots z_d^{v-1} \phi_d(1).$$

Consequently, from Lemma 2.1(ii) we get

$$\psi_d(1) - z_1^{v-1} \cdots z_d^{v-1} \phi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) = \psi_d(1) - {}_v\phi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) \in (z_1^v, \dots, z_d^v) \text{sy}z_d^{F_\bullet}(Q/\mathfrak{m}).$$

This, in conjunction with the CE Property assumed in the statement implies that $\psi_d = g'_d \circ g_d \neq 0$, whence $g_d \neq 0$ in particular. Consequently, $t \geq d$, otherwise the target of $g_d : K_d(\mathbf{z}^v; Q) \rightarrow P_d$ is the zero module (we recall that the complex P_\bullet has length t , where $t \leq s + \ell$). So $\ell + s \geq d$. Therefore,

$$\dim(Q/\mathbf{x}Q) = \dim(Q/\mathfrak{m}/\mathbf{x}Q/\mathfrak{m}) = \dim(Q/\mathfrak{m}) - \ell = \text{height}(\mathfrak{m}) - \ell = d - \ell \leq s,$$

as was to be proved.

(ii) Suppose that S is a homomorphic image of a Gorenstein local ring G , say $S = G/\mathfrak{g}$ where \mathfrak{g} is an ideal of G of height l . From our hypothesis in conjunction with [Ho83, Theorem (3.15)] we get $\eta_{S/xS} \neq 0$. We are done if we can show that $\eta_S \neq 0$ (again by [Ho83, Theorem (3.15)]). Since $l = \dim(G) - \dim(S)$ in view of [BrHe98, Corollary 2.1.4], so by virtue of [Ho83, Theorem (5.3)] the non-vanishing of η_S follows from the vanishing of the module

$$C = 0 :_{\text{Ext}_G^{l+1}(S, G)} x$$

in the statement of [Ho83, Theorem (5.3)], where C is introduced in [Ho83, Lemma (5.1) and its preceding paragraph].

By [BrSh13, 11.2.6 Local Duality Theorem]), $\text{Hom}_S(\text{Ext}_G^{l+1}(S, G), \text{E}(S/\mathfrak{m}_S)) \cong H_{\mathfrak{m}_S}^{\dim(S)-1}(S)$ which implies that

$$\begin{aligned} \text{Hom}_S(C, \text{E}(S/\mathfrak{m}_S)) &\cong \text{Hom}_S\left(\text{Hom}_S(S/xS, \text{Ext}_G^{l+1}(S, G)), \text{E}(S/\mathfrak{m}_S)\right) \\ &\cong \text{Hom}_S(\text{Ext}_G^{l+1}(S, G), \text{E}(S/\mathfrak{m}_S)) \otimes_S (S/xS) && \text{([Ro09, Lemma 9.71])} \\ &\cong H_{\mathfrak{m}_S}^{\dim(S)-1}(S)/xH_{\mathfrak{m}_S}^{\dim(S)-1}(S) \\ &= 0 && \text{(our hypo. and Lemma 2.11(iii)).} \end{aligned}$$

Consequently $C = 0$, as required. \square

Lemma 2.21. *Let $\sigma : (S, \mathfrak{m}_S) \rightarrow (T, \mathfrak{m}_T)$ be a local homomorphism of local rings such that the image of a system of parameters for S is a system of parameters for T . If T satisfies the CE Property, then so does S .*

Proof. See [Ho83, Proposition (3.18)(a)] and [Ho83, Theorem (3.15)]. \square

3. EXTENDED REES ALGEBRA OF PRIME ALMOST COMPLETE INTERSECTION IDEALS

This section is devoted to proving Proposition 3.6 on some nice properties of the extended Rees algebra $A[\mathfrak{a}t, t^{-1}]$, where \mathfrak{a} is a prime almost complete intersection ideal of a regular local ring (A, \mathfrak{m}_A) . The following lemmas are required for Proposition 3.6.

Lemma 3.1. *Let (A, \mathfrak{m}_A) be a regular local ring and $\mathfrak{b} := \mathfrak{c} : h$, $\mathfrak{a} := (\mathfrak{c}, h)$ be linked ideals over the complete intersection ideal $\mathfrak{c} := (a_1, \dots, a_g)$ such that \mathfrak{b} is a quasi-Gorenstein prime ideal, and \mathfrak{a} is a prime almost complete intersection ideal. Assume that A/\mathfrak{b} is not Cohen-Macaulay. Then*

- (i) $\text{depth}(A/\mathfrak{a}) = \text{depth}(A/\mathfrak{b}) - 1$.
- (ii) a_1, \dots, a_g, h is a d -sequence.
- (iii) *there exists a regular sequence \mathbf{x} in A whose image in A/\mathfrak{c} is a regular sequence, whose image in A/\mathfrak{a} is a maximal regular sequence, and such that the image of \mathbf{x}, h in A/\mathfrak{b} is a maximal regular sequence.*

Proof. (i) From the exact sequence $0 \rightarrow A/\mathfrak{b} \xrightarrow{h} A/\mathfrak{c} \rightarrow A/\mathfrak{a} \rightarrow 0$, we conclude that $\text{depth}(A/\mathfrak{a}) = \text{depth}(A/\mathfrak{b}) - 1$. Since A/\mathfrak{c} is a complete intersection (thus Cohen-Macaulay) and A/\mathfrak{b} is non-Cohen-Macaulay, so the statement follows easily from the long exact sequence obtained by $\Gamma_{\mathfrak{m}_A}(-)$ and the above exact sequence.

(ii) We only need to show that $\mathfrak{c} : h^i = \mathfrak{c} : h$ for all $i \geq 2$, as we already know that a_1, \dots, a_g is a regular sequence. But $\mathfrak{c} : h$ is a codimension g prime quasi-Gorenstein ideal by our hypothesis while $\mathfrak{c} : h^i$ contains $\mathfrak{c} : h$, so we are done once we can prove that $\mathfrak{c} : h^i$ also has codimension g . If $\text{height}(\mathfrak{c} : h^i) \geq g+1$ then $\mathfrak{c} : h^i \not\subseteq \bigcup_{\mathfrak{p} \in \text{ass}(\mathfrak{c})} \mathfrak{p}$, because \mathfrak{c} is a complete intersection thus an unmixed ideal. It follows that then

there exists some $x \in A$ such that (a_1, \dots, a_g, x) is a regular sequence while $x \in \mathfrak{c} : h^i$. But then $h^i \in \mathfrak{c}$ implying that $\mathfrak{a}^i \subseteq \mathfrak{c} \subseteq \mathfrak{b}$, whence $\mathfrak{a} = \mathfrak{b}$ (\mathfrak{a} is also a codimension g prime ideal). But this is impossible because quasi-Gorenstein ideals are never almost complete intersection by [Ku74, Proposition 1.1].

(iii) The element h is regular on A/\mathfrak{b} and since $\text{depth } A/\mathfrak{a} = \text{depth } A/(\mathfrak{b}, h) < \text{depth } A/\mathfrak{c} < \text{depth } A$, the statement is a simple result of the prime avoidance lemma (similar to what will be done in Step 4 of the proof of Theorem 5.1). \square

The goal of the next lemma is to illustrate our situation in the last step of the proof of Theorem 5.1.

Lemma 3.2. *Suppose that $\mathfrak{a} = (h, a_1, \dots, a_g)$ is an ideal of a ring A .*

(i) *Let $A[Z_0, \dots, Z_g]$ be the standard polynomial ring over A . The symmetric algebra of \mathfrak{a} , denoted by $\text{Sym}_A(\mathfrak{a})$, as a graded R -algebra is isomorphic to the quotient ring*

$$A[Z_0, \dots, Z_g]/\mathfrak{d}, \quad \text{where } \mathfrak{d} = \langle \{r_0 Z_0 + \sum_{i=1}^g r_i Z_i : r_0, \dots, r_g \in A, hr_0 + \sum_{i=1}^g r_i a_i = 0\} \rangle.$$

(ii) *We denote the sequence of degree 1 homogeneous elements $h, a_1, \dots, a_g \in \mathfrak{a} = \text{Sym}_A(\mathfrak{a})_{[1]}$ by $ht, a_1 t, \dots, a_g t$. Then*

$$A/((a_1, \dots, a_g) : h) \cong (\text{Sym}_A(\mathfrak{a})_{ht})_{[0]}/(a_1 t/ht, \dots, a_g t/ht).$$

Proof. (i) This is a folklore fact in the literature. The statement is, e.g., written in [Hu80, page 270] without proof.

(ii) The dehomogenization with respect to Z_0

(3.1)

$$A[Z_0, \dots, Z_g]/(\mathfrak{d}, Z_0 - 1) \rightarrow ((A[Z_0, \dots, Z_g]/\mathfrak{d})_{Z_0})_{[0]}, \quad (\text{induced by } Z_i \mapsto (Z_i + \mathfrak{d})/Z_0, r \in A \mapsto (r + \mathfrak{d})/1)$$

is an isomorphism. For the injectivity of this ring homomorphism, suppose that $\sum_{i=1}^t f_i + (\mathfrak{d}, Z_0 - 1)$ belongs to the kernel where each f_i is a non-zero monomial of degree n_i . Then, there exists $m \in \mathbb{N}_0$ with $\sum_{i=1}^t (Z_0^{m-n_i} f_i) \in \mathfrak{d}$. Since Z_0 is treated as 1 in $A[Z_0, \dots, Z_g]/(\mathfrak{d}, Z_0 - 1)$ so the injectivity follows.

The isomorphism (3.1) and part (i) yield

$$A[Z_0, \dots, Z_g]/(\mathfrak{d}, Z_0 - 1) \cong (\text{Sym}_A(\mathfrak{a})_{ht})_{[0]}, \quad (\text{induced by } Z_i \mapsto a_i t/ht, Z_0 \rightarrow 1, r \in A \mapsto r).$$

Consequently,

$$\begin{aligned} A/((a_1, \dots, a_g) : h) &\cong A[Z_0, \dots, Z_g]/(\mathfrak{d}, Z_0 - 1, Z_1, \dots, Z_g) && (\text{by the definition of } \mathfrak{d}) \\ &\cong (\text{Sym}_A(\mathfrak{a})_{ht})_{[0]}/(a_1 t/ht, \dots, a_g t/ht) && (\text{the previous display}). \quad \square \end{aligned}$$

Definition 3.3. We say that a ring S is a *locally complete intersection at codimension ≤ 1* precisely when $S_{\mathfrak{p}}$ is a complete intersection ring for each prime ideal \mathfrak{p} of S with $\text{height}(\mathfrak{p}) \leq 1$.

The next lemma will be used in the proof of Proposition 3.6(iv).

Lemma 3.4. *(due to Raymond Heitmann) Let A be a ring and suppose that a_1, \dots, a_g is a regular sequence of A . Let $\mathfrak{A} := (a_1, \dots, a_g, h)A$ and suppose that $((a_1, \dots, a_g) : h^i) = ((a_1, \dots, a_g) : h)$ for all $i \geq 2$. Then for every j , $1 \leq j \leq g$ and every positive integer n , $(a_1, \dots, a_j)A \cap \mathfrak{A}^n = (a_1, \dots, a_j)\mathfrak{A}^{n-1}$.*

Proof. We first prove the $j = g$ case for all n . Obviously, $\mathfrak{A}^n = (a_1, \dots, a_g)\mathfrak{A}^{n-1} + h^n A$ and so it suffices to show that whenever $h^n w \in (a_1, \dots, a_g)$, $h^n w \in (a_1, \dots, a_g)\mathfrak{A}^{n-1}$. However, by the hypothesis, $hw \in (a_1, \dots, a_g)$ and so $h^n w \in h^{n-1}(a_1, \dots, a_g) \subseteq (a_1, \dots, a_g)\mathfrak{A}^{n-1}$.

Now, we handle the $j < g$ case by induction on n . The $n = 1$ case holds trivially. So let $n \geq 2$. Suppose that $w \in (a_1, \dots, a_j) \cap \mathfrak{A}^n$. By the $j = g$ case, we have $w \in (a_1, \dots, a_g)\mathfrak{A}^{n-1}$. We can choose

$u \leq g$ minimal so that $w \in (a_1, \dots, a_u)\mathfrak{A}^{n-1}$. We finish the proof by induction on u . If $u \leq j$, we are done. Otherwise, we write $w = w' + a_u\alpha$ with $w' \in (a_1, \dots, a_{u-1})\mathfrak{A}^{n-1}$ and $\alpha \in \mathfrak{A}^{n-1}$. Since a_1, \dots, a_u is a regular sequence, $\alpha \in (a_1, \dots, a_{u-1}) \cap \mathfrak{A}^{n-1}$. By the induction assumption on n , we have $\alpha \in (a_1, \dots, a_{u-1})\mathfrak{A}^{n-2}$ and so $a_u\alpha \in (a_1, \dots, a_{u-1})\mathfrak{A}^{n-1}$. By induction on u , the proof is complete. \square

Remark 3.5. The preceding lemma and Proposition 3.6(ii) are pointed out to the author by Raymond Heitmann and the referee to simplify dramatically the (previous) author's proof of Proposition 3.6(iv) and (v) where the author had exploited Herzog-Simis-Vasconcelos \mathcal{M} -approximation complexes through a spectral sequence argument. We are thankful to Raymond Heitmann and the referee for their favor.

The extended Rees algebra $A[\mathfrak{a}t, t^{-1}]$ in the next proposition is a factorial domain. This factoriality follows from the hypothesis that A/\mathfrak{a} is a locally complete intersection in codimension ≤ 1 , by virtue of [Hoc73, Theorem 1] (see parts F and A) and [Hu81, Theorem 2.2].

Proposition 3.6. *Let (A, \mathfrak{m}_A) be a regular local ring of dimension d and $\mathfrak{a} := (a_1, \dots, a_g, h)$ be a prime almost complete intersection ideal of A , containing the codimension g complete intersection ideal $\mathfrak{c} := (a_1, \dots, a_g)$. Suppose further that the quasi-Gorenstein ideal $\mathfrak{b} := (a_1, \dots, a_g) : h$, which is linked to \mathfrak{a} , is a prime ideal such that A/\mathfrak{b} is not Cohen-Macaulay. Denote by \mathfrak{M} the unique homogeneous maximal ideal of $A[\mathfrak{a}t, t^{-1}]$. To force $A[\mathfrak{a}t, t^{-1}]$ to be a factorial domain, we assume that A/\mathfrak{a} is a locally complete intersection at codimension ≤ 1 . Then the factorial extended Rees algebra $A[\mathfrak{a}t, t^{-1}]$ satisfies the following properties.*

(i) $\mathfrak{P} := (\mathfrak{m}_A, a_1t, \dots, a_gt, t^{-1})$ is a height d prime ideal.

(ii) Let $(A/\mathfrak{c})[(h)s, s^{-1}]$ be the extended Rees algebra of $h(A/\mathfrak{c})$. Then

$$(A/\mathfrak{c})[(h)s, s^{-1}] \cong A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt).$$

(iii) $(A/\mathfrak{a})[X] \cong A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt, ht)^2$.

(iv) Let $\mathfrak{x} := x_1, \dots, x_u \subseteq A$ be the regular sequence of A obtained in Lemma 3.1(iii), that forms a maximal regular sequence on the almost complete intersection quotient domain A/\mathfrak{a} and a regular sequence on the rings $A/\mathfrak{b}, A/\mathfrak{c}$. Then

$$t^{-1}, a_1t, \dots, a_gt, x_1, \dots, x_u$$

(and any of its permutation) is a regular sequence on $A[\mathfrak{a}t, t^{-1}]$. It is maximal with respect to the property of consisting of only homogeneous elements.

(v) For each $0 \leq i \leq g-1$,

$$a_{i+1}t \notin \bigcup_{\mathfrak{p} \in \text{Att}\left(H_{\mathfrak{M}}^{d-i}(A[\mathfrak{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_it))\right)} \mathfrak{p}.$$

Proof. (i) We have a homogeneous ring epimorphism $\pi : (A/\mathfrak{m}_A)[X] \xrightarrow{X \mapsto ht} A[\mathfrak{a}t, t^{-1}]/\mathfrak{P}$. Since $(A/\mathfrak{m}_A)[X]$ is a principal ideal domain and π is homogeneous so either π is an isomorphism or $\ker(\pi) = (X^i)$ for some $i \in \mathbb{N}$. We show that the latter is impossible. If it were possible, then $(ht)^i \in \mathfrak{P} = (a_1t, \dots, a_gt, \mathfrak{m}_A, t^{-1})$ which implies that

$$h^i \in \mathfrak{m}_A \mathfrak{a}^i + \sum_{j=1}^g a_j \mathfrak{a}^{i-1} + \mathfrak{a}^{i+1} \subseteq (\mathfrak{c}, \mathfrak{m}_A h^i, h^{i+1}) \subseteq (\mathfrak{c}, \mathfrak{m}_A h^i).$$

²We stress that the ideal (a_1t, \dots, a_gt, ht) is not a complete intersection, but it is generated by $g+1$ elements while it contains the length g complete intersection (a_1t, \dots, a_gt) .

It follows that $(1+m)h^i \in \mathfrak{c}$ for some $m \in \mathfrak{m}_A$, i.e. $h^i \in \mathfrak{c}$. But this contradicts the fact that

$$\mathfrak{c} : h^i \stackrel{\text{Lemma 3.1(ii)}}{=} \mathfrak{c} : h = \mathfrak{b}$$

is our (quasi-Gorenstein) prime ideal.

(ii) It is obvious that $(A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt))_{[0]} = A/\mathfrak{c}$. Set $\mathfrak{a}^i = A$ for any $i \leq 0$. The identities

$$\forall i \in \mathbb{Z}, \quad (A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt))_{[i]} = \mathfrak{a}^i/\mathfrak{c}\mathfrak{a}^{i-1} \stackrel{\text{Lemma 3.4}}{=} \mathfrak{a}^i/(\mathfrak{c} \cap \mathfrak{a}^i),$$

provide us with the family of isomorphisms

$$\varphi_i : (A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt))_{[i]} \rightarrow h^i(A/\mathfrak{c}), \quad h^i + (\mathfrak{c}\mathfrak{a}^{i-1}) \mapsto h^i + \mathfrak{c}.$$

Evidently, the maps φ_i yield a homogeneous ring isomorphism $A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt) \rightarrow (A/\mathfrak{c})[(h)s, s^{-1}]$ as claimed in the statement.

(iii) This isomorphism is easy, because the natural ring homomorphism $A \rightarrow A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt, ht)$ induces a homogeneous ring epimorphism

$$(A/\mathfrak{a})[X] \rightarrow A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt, ht), \quad X \mapsto t^{-1}$$

and to see that this is injective, it suffices to note that each homogeneous element in the kernel is zero.

(iv) Applying Lemma 3.4, it is straightforward to observe that $t^{-1}, a_1t, \dots, a_gt$ is a regular sequence on $A[\mathfrak{a}t, t^{-1}]$. More immediately, in light of Lemma 3.4 and [HuMa97, Proposition 3.5] the sequence a_1t, \dots, a_gt forms a regular sequence on $\text{gr}_{\mathfrak{a}}(A) = A[\mathfrak{a}t, t^{-1}]/(t^{-1})$. Hence, we should show that \mathbf{x} forms a regular sequence on

$$A[\mathfrak{a}t, t^{-1}]/(t^{-1}, a_1t, \dots, a_gt) = (A/\mathfrak{c})[(h)s, s^{-1}]/(s^{-1}) = \text{gr}_{h(A/\mathfrak{c})}(A/\mathfrak{c}).$$

But for each $i \geq 1$ there is an isomorphism $A/(\mathfrak{b}, h) \xrightarrow{1 \mapsto h^i} (h^i, \mathfrak{c})/(h^{i+1}, \mathfrak{c})$ (by Lemma 3.1(ii)). Therefore

$$(3.2) \quad \text{gr}_{h(A/\mathfrak{c})}(A/\mathfrak{c}) = A/\mathfrak{a} \oplus \left(\bigoplus_{i=1}^{\infty} A/(\mathfrak{b}, h) \right)$$

as A/\mathfrak{c} -modules, so the sequence \mathbf{x} is regular on $\text{gr}_{A/\mathfrak{c}}(h(A/\mathfrak{c}))$ as needed.

It is proved that the desired sequence is regular on $A[\mathfrak{a}t, t^{-1}]$. To see that this is a maximal homogeneous regular sequence, note that the quotient of $A[\mathfrak{a}t, t^{-1}]$ by this regular sequence is generated by only one element of degree 1 that is the residue class of ht . So if we can extend this regular sequence by adding a homogeneous element, the new element would be either some power of ht or an element of degree zero. The latter is impossible, because the degree zero subring of $A[\mathfrak{a}t, t^{-1}]/(t^{-1}, a_1t, \dots, a_gt, x_1, \dots, x_u)$ is the almost complete intersection $A/(\mathfrak{a} + (\mathbf{x}))$ which has depth zero by our choice of \mathbf{x} . The former case is also impossible, otherwise $t^{-1}, a_1t, \dots, a_gt, ht$ would be a regular sequence of $A[\mathfrak{a}t, t^{-1}]$ which contradicts with $A[\mathfrak{a}t, t^{-1}]/(t^{-1}, a_1t, \dots, a_gt, ht) = A/\mathfrak{a}$ (part (iii)), because then $A[\mathfrak{a}t, t^{-1}]_{\mathfrak{M}}$, as a deformation of an almost complete intersection, would be an almost complete intersection by [BrHe98, Theorem 2.3.4(a)] and [Ma89, Theorem 21.1(iii)]. However, almost complete intersections are not quasi-Gorenstein ([Ku74, Proposition 1.1]) while $A[\mathfrak{a}t, t^{-1}]_{\mathfrak{M}}$ is a factorial domain that is a homomorphic image of a regular local ring and so it is quasi-Gorenstein (see e.g. [FoFoGrRe75, Lemma (2.4)]).

(v) First we show that $Q := A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_gt)$ satisfies the (S_2) -condition. Let $\mathfrak{p} \in \text{Spec}(A[\mathfrak{a}t, t^{-1}])$ such that it contains (a_1t, \dots, a_gt) . If $t^{-1} \notin \mathfrak{p}$, then $A[\mathfrak{a}t, t^{-1}]$ is regular at \mathfrak{p} due to the regularity of $A[\mathfrak{a}t, t^{-1}]_{t^{-1}} \cong A[t, t^{-1}]$, whence Q is a complete intersection at \mathfrak{p} by part (iii).

If $t^{-1} \in \mathfrak{p}$, then we are done once we can show that

$$G := \text{gr}_{\mathfrak{a}}(A)/(a_1 + \mathfrak{a}^2, \dots, a_g + \mathfrak{a}^2) \cong Q/(t^{-1})$$

satisfies the (S_1) -condition (again t^{-1} is a regular element of Q in view of part (iv)), because then $\text{depth}(G_{\mathfrak{p}}) \geq \min\{\dim(G_{\mathfrak{p}}), 1\}$ implying that $\text{depth}(Q_{\mathfrak{p}}) \geq \min\{\dim(Q_{\mathfrak{p}}), 2\}$, as required. In the rest of the proof, by abuse of notation, we consider \mathfrak{p} as a prime ideal of G . We also denote the image in G , or in any quotient of A , of any element $r \in A$ by the same notation r .

If $ht \notin \mathfrak{p}$, then by part (ii) (and Lemma 3.1(ii) for the second isomorphism)

$$\begin{aligned} Q_{ht} &\cong \left((A/\mathfrak{c})[(h)s, s^{-1}] \right)_{hs} \\ &\cong \left((A/\mathfrak{c})[X, Y]/(XY - h, (0 :_{A/\mathfrak{c}} h)X) \right)_X, & (X \mapsto hs, Y \mapsto s^{-1}) \\ &\cong \left((A/\mathfrak{c})[X, Y]/(XY - h, 0 :_{A/\mathfrak{c}} h) \right)_X \\ &\cong \left((A/\mathfrak{b})[X, Y]/(XY - h) \right)_X. \end{aligned}$$

Thus

$$G_{ht} = (Q/(t^{-1}))_{ht} \cong \left((A/\mathfrak{b})[X, Y]/(XY - h, Y) \right)_X \cong \left((A/(\mathfrak{b}, h))[X] \right)_X$$

which is indeed (S_1) because A/\mathfrak{b} is an (S_2) -domain.

It remains only to discuss the case where $t^{-1}, ht \in \mathfrak{p}$. Let $\mathfrak{q} := \mathfrak{p} \cap A/\mathfrak{a} = \mathfrak{p} \cap G_{[0]}$. We can assume furthermore that $\mathfrak{a} \subsetneq \mathfrak{q}$, because otherwise $\text{height}_G(\mathfrak{p}) = 0$ necessarily and then there is nothing to prove. Let $r \in \mathfrak{q} \setminus \mathfrak{a}$. If r does not belong to any associated prime (or equivalently minimal prime) of (\mathfrak{b}, h) , then r is regular on $G = \text{gr}_{h(A/\mathfrak{c})}(A/\mathfrak{c})$ in view of display (3.2) and thus \mathfrak{p} contains a regular element as needed. If otherwise, there exists some $s \in A \setminus (\mathfrak{b}, h)$ such that $sr \in (\mathfrak{b}, h)$. Note that $sr \notin \mathfrak{a}$ because \mathfrak{a} is a prime ideal. Then, there is some $\beta \in \mathfrak{b} \setminus \mathfrak{a}$ such that $sr + \mathfrak{a} = \beta + \mathfrak{a}$, thus $\beta \in (\mathfrak{q} \cap \mathfrak{b}) \setminus \mathfrak{a}$. Consequently, $\beta + ht \in \mathfrak{p}$ (here the image of ht in $G = Q/(t^{-1})$ is denoted by the same notation ht). Then it is readily seen that $\beta + ht \in \mathfrak{p}$ is a regular element of G (because \mathfrak{a} is a prime ideal, $0 :_G ht = 0 :_G (ht)^2$ as can easily be verified and $\beta(ht) = 0$ in G).

It follows that $Q = A[\mathfrak{a}t, t^{-1}]/(a_1t, \dots, a_g t)$ satisfies the (S_2) -condition, as was to be proved. Hence in light of [Gr65, Corollaire (5.12.4)], each $A[\mathfrak{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_it)$ satisfies (S_2) -condition ($1 \leq i \leq g$). So our statement follows from the equivalent conditions of [TaTu18, Corollary 2.8] (again we are using the fact that $A[\mathfrak{a}t, t^{-1}]_{\mathfrak{M}}$ is a factorial domain and quasi-Gorenstein). \square

4. A NEW VARIANT OF THE CANONICAL ELEMENT THEOREM

4.1. Various aspects of our new variant of the Canonical Element Theorem. We begin this section by recalling the definition of the canonical element with respect to a primary ideal to the maximal ideal.

Definition 4.1. (see [Ho83, Definition 3.2]) Let (S, \mathfrak{m}_S) be a d -dimensional local ring, \mathfrak{q} be an \mathfrak{m}_S -primary ideal and F_{\bullet} be a free resolution of S/\mathfrak{q} . Consider the natural epimorphism $\pi : F_d \rightarrow \text{syz}_d^{F_{\bullet}}(S/\mathfrak{q})$ defining the element

$$\epsilon_{\mathfrak{q}} := [\pi] \in \text{Ext}_S^d(S/\mathfrak{q}, \text{syz}_d^{F_{\bullet}}(S/\mathfrak{q})).$$

Let $\eta_{\mathfrak{q}}$ be the image of $\epsilon_{\mathfrak{q}}$ under the natural map to the local cohomology

$$\Theta_{\mathfrak{q}}^{\text{syz}_d^{F_{\bullet}}(S/\mathfrak{q})} : \text{Ext}_S^d(S/\mathfrak{q}, \text{syz}_d^{F_{\bullet}}(S/\mathfrak{q})) \rightarrow \varinjlim_{m \in \mathbb{N}} \text{Ext}_S^d(S/\mathfrak{q}^m, \text{syz}_d^{F_{\bullet}}(S/\mathfrak{q})).$$

Then $\eta_{\mathfrak{q}}$ is said to be the *canonical element* of S with respect to \mathfrak{q} . The element $\eta_{\mathfrak{m}_S}$ is called the *canonical element* of S and it is denoted, also, by η_S . We recall that for any $s \in S$, the vanishing/non-vanishing of $s \cdot \epsilon_{\mathfrak{q}}$ and $s \cdot \eta_{\mathfrak{q}}$ is independent of the chosen resolution F_{\bullet} . ([Ho83, Proposition (3.3)(c)]).

Remark 4.2. Let \mathfrak{p} be a prime ideal of a ring S and $y \in S \setminus \mathfrak{p}$ be a non-unit. If F_\bullet is the minimal free resolution of S/\mathfrak{p} , then the mapping cone

$${}_wG_\bullet := \text{cone}(y^w \cdot \text{id}_{F_\bullet} : F_\bullet \rightarrow F_\bullet)$$

provides us with the minimal free resolution of $S/(\mathfrak{p}, y^w)$. This fact can be observed via the exact sequence of complexes

$$(4.1) \quad 0 \rightarrow F_\bullet \rightarrow {}_wG_\bullet \rightarrow F_\bullet[-1] \rightarrow 0$$

inducing the homology long exact sequence

$$\cdots \rightarrow H_i(F_\bullet) \xrightarrow{y^w} H_i(F_\bullet) \rightarrow H_i({}_wG_\bullet) \rightarrow H_{i-1}(F_\bullet) \xrightarrow{y^w} H_{i-1}(F_\bullet) \rightarrow \cdots$$

as y^w forms a non-zero divisor over the integral domain S/\mathfrak{p} (see [Ro09, Lemma 10.38] for a reference to these exact sequences).

In the statement of the following lemma, all syzygies are obtained from the minimal free resolutions. So when we are denoting a syzygy in the lemma, we will omit the particular free resolution by which the syzygy is computed.

Lemma 4.3. *Let \mathfrak{p} be a prime ideal of a ring S , $y \in S \setminus \mathfrak{p}$ be a non-unit and $i, s, t \in \mathbb{N}$ with $t \geq s$ (s and t stand for powers of y , while i stands for the i -th syzygy). Consider the minimal free resolutions ${}_sG_\bullet$ and ${}_tG_\bullet$ of $S/(\mathfrak{p}, y^s)$ and $S/(\mathfrak{p}, y^t)$ respectively, as mentioned in Remark 4.2.*

(i) *There are chain maps*

$${}_{t,s}\lambda_\bullet : {}_tG_\bullet \rightarrow {}_sG_\bullet, \quad {}_{s,t}\lambda_\bullet : {}_sG_\bullet \rightarrow {}_tG_\bullet$$

inducing S -homomorphisms respectively

$$f_{t,s} : \text{syzy}_i(S/(\mathfrak{p}, y^t)) \rightarrow \text{syzy}_i(S/(\mathfrak{p}, y^s)), \quad f_{s,t} : \text{syzy}_i(S/(\mathfrak{p}, y^s)) \rightarrow \text{syzy}_i(S/(\mathfrak{p}, y^t))$$

such that

$$f_{s,t} \circ f_{t,s} = y^{t-s} \text{id}_{\text{syzy}_i(S/(\mathfrak{p}, y^t))}, \quad {}_{s,t}\lambda_\bullet \circ {}_{t,s}\lambda_\bullet = y^{t-s} \text{id}_{{}_tG_\bullet}, \quad {}_{t,s}\lambda_0 = \text{id}_S.$$

(ii) *If y is a regular element of S , then there is an exact sequence*

$$0 \rightarrow \text{syzy}_i(S/(\mathfrak{p}, y^t)) \xrightarrow{f_{t,s}} \text{syzy}_i(S/(\mathfrak{p}, y^s)) \rightarrow \text{syzy}_{i-1}(S/\mathfrak{p})/y^{t-s} \text{syzy}_{i-1}(S/\mathfrak{p}) \rightarrow 0.$$

(iii) *If y is a regular element of S , then there is an exact sequence*

$$0 \rightarrow \text{syzy}_i(S/(\mathfrak{p}, y^s)) \xrightarrow{f_{s,t}} \text{syzy}_i(S/(\mathfrak{p}, y^t)) \rightarrow \text{syzy}_i(S/\mathfrak{p})/y^{t-s} \text{syzy}_i(S/\mathfrak{p}) \rightarrow 0.$$

Proof. In view of the rules of the differentials of ${}_sG_\bullet$ and ${}_tG_\bullet$, there is a chain map ${}_{t,s}\lambda_\bullet : {}_tG_\bullet \rightarrow {}_sG_\bullet$ given by

$$\begin{pmatrix} \text{id}_{F_j} & 0 \\ 0 & y^{t-s} \text{id}_{F_{j-1}} \end{pmatrix} : \underbrace{F_j \oplus F_{j-1}}_{{}_tG_j} \rightarrow \underbrace{F_j \oplus F_{j-1}}_{{}_sG_j}$$

for each j . By definition, ${}_{t,s}\lambda_0 = \text{id}_S$. This chain map induces $f_{t,s} : \text{syzy}_i(S/(\mathfrak{p}, y^t)) \rightarrow \text{syzy}_i(S/(\mathfrak{p}, y^s))$ which fits in the diagram with exact rows of S -modules and homomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{syzy}_i(S/\mathfrak{p}) & \longrightarrow & \text{syzy}_i(S/(\mathfrak{p}, y^t)) & \longrightarrow & \text{syzy}_{i-1}(S/\mathfrak{p}) \longrightarrow 0 \\ & & \parallel & & \downarrow f_{t,s} & & \downarrow y^{t-s} \text{id}_{\text{syzy}_{i-1}(S/\mathfrak{p})} \\ 0 & \longrightarrow & \text{syzy}_i(S/\mathfrak{p}) & \longrightarrow & \text{syzy}_i(S/(\mathfrak{p}, y^s)) & \longrightarrow & \text{syzy}_{i-1}(S/\mathfrak{p}) \longrightarrow 0 \end{array}$$

where the exact sequence of rows exist in view of display (4.1) in the previous remark. Now, part (ii) follows after applying the Snake Lemma to the above diagram.

Dual to the above chain map, one can see that there is a chain map ${}_{s,t}\lambda_\bullet : {}_sG_\bullet \rightarrow {}_tG_\bullet$ given by

$$\begin{pmatrix} y^{t-s}\text{id}_{F_j} & 0 \\ 0 & \text{id}_{F_{j-1}} \end{pmatrix} : \underbrace{F_j \oplus F_{j-1}}_{{}_sG_j} \rightarrow \underbrace{F_j \oplus F_{j-1}}_{{}_tG_j}$$

which similarly as above induces $f_{s,t} : \text{syz}_i(S/(\mathfrak{p}, y^s)) \rightarrow \text{syz}_i(S/(\mathfrak{p}, y^t))$ fitting in the diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{syz}_i(S/\mathfrak{p}) & \longrightarrow & \text{syz}_i(S/(\mathfrak{p}, y^s)) & \longrightarrow & \text{syz}_{i-1}(S/\mathfrak{p}) \longrightarrow 0 \\ & & \downarrow y^{t-s}\text{id}_{\text{syz}_i(S/\mathfrak{p})} & & \downarrow f_{s,t} & & \parallel \\ 0 & \longrightarrow & \text{syz}_i(S/\mathfrak{p}) & \longrightarrow & \text{syz}_i(S/(\mathfrak{p}, y^t)) & \longrightarrow & \text{syz}_{i-1}(S/\mathfrak{p}) \longrightarrow 0 \end{array} .$$

Again part (iii) follows from the Snake Lemma and the above diagram. The last statement of part (i) is evident from the definition of the aforementioned chain maps. \square

The next proposition is, roughly speaking, a part of Proposition-Definition 4.6.

Proposition 4.4. *Let (S, \mathfrak{m}_S) be a local ring of dimension d which is a homomorphic image of a Gorenstein ring (thus it admits a canonical module ω_S). Let \mathfrak{q} be an ideal of S of height $d-1$ and x_1 be an element of S such that (\mathfrak{q}, x_1) is \mathfrak{m}_S -primary. The following statements are equivalent (for the notation used here see Definition 4.1):*

(i) $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)} \neq 0$.

(ii) Let $\nu_w : S/(\mathfrak{q}, x_1) \rightarrow S/(\mathfrak{q}, x_1^w)$ be the multiplication map by x_1^{w-1} . The following map is non-zero:

$$\text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) \xrightarrow{\text{Ext}_S^d(\nu_w, \text{id}_{\omega_S})} \text{Ext}_S^d(S/(\mathfrak{q}, x_1), \omega_S) \xrightarrow{\Theta_{(\mathfrak{q}, x_1)}^{\omega_S}} H_{(\mathfrak{q}, x_1)}^d(\omega_S).$$

(iii) The following map is non-zero:

$$\text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) \xrightarrow{\Theta_{(\mathfrak{q}, x_1^w)}^{\omega_S}} H_{(\mathfrak{q}, x_1^w)}^d(\omega_S) \xrightarrow{x_1^{w-1}} H_{(\mathfrak{q}, x_1^w)}^d(\omega_S).$$

Proof. (i) \Leftrightarrow (ii): Our proof is an adaptation of the proof of [Ho83, Theorem (4.3)] to the situation of our lemma. Let

$$\Psi_M : H_{(\mathfrak{q}, x_1)}^d(M) \rightarrow H_{(\mathfrak{q}, x_1^w)}^d(M)$$

be the isomorphism induced by the maps $\text{Ext}_S^d(S/(\mathfrak{q}, x_1)^n, M) \rightarrow \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w)^n, M)$ each of which, itself, is induced by the natural epimorphism $\pi_n : S/(\mathfrak{q}, x_1^w)^n \rightarrow S/(\mathfrak{q}, x_1)^n$. Let us also to denote Ψ_{ω_S} by Ψ . First we prove two claims.

Claim 1: $\Psi_M \circ \Theta_{(\mathfrak{q}, x_1)}^M \circ \text{Ext}_S^d(\nu_w, \text{id}_M) = x_1^{w-1} \cdot \Theta_{(\mathfrak{q}, x_1^w)}^M$ for each S -module M .

Proof of the claim: Our claim easily follows from the identities

$$\begin{aligned} \Psi_M \circ \Theta_{(\mathfrak{q}, x_1)}^M \circ \text{Ext}_S^d(\nu_w, \text{id}_M) &= \Theta_{(\mathfrak{q}, x_1^w)}^M \circ \text{Ext}_S^d(\pi_1, \text{id}_M) \circ \text{Ext}_S^d(\nu_w, \text{id}_M) \\ &= \Theta_{(\mathfrak{q}, x_1^w)}^M \circ \text{Ext}_S^d(x_1^{w-1}\text{id}_{S/(\mathfrak{q}, x_1^w)}, \text{id}_M). \end{aligned}$$

Claim 2: If there is some S -module M for which the map

$$\Theta_{(\mathfrak{q}, x_1)}^M \circ \text{Ext}_S^d(\nu_w, \text{id}_M) : \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), M) \rightarrow H_{(\mathfrak{q}, x_1)}^d(M)$$

is non-zero, then $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)} \neq 0$.

Proof of the claim: Let $(G_\bullet, \partial_\bullet^G)$ be a free resolution of $S/(\mathfrak{q}, x_1^w)$, and pick some $\varphi : G_d \rightarrow M$ such that $[\varphi] \in \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), M)$ has non-zero image under $\Theta_{(\mathfrak{q}, x_1)}^M \circ \text{Ext}_S^d(\nu_w, \text{id}_M)$. As φ is a cycle, φ factors through $\overline{\varphi} : G_d/\text{im}(\partial_{d+1}^G) = \text{syz}_d^G(S/(\mathfrak{q}, x_1^w)) \rightarrow M$. By our hypothesis and the bijectivity of Ψ_M ,

$$(4.2) \quad x_1^{w-1} \cdot \Theta_{(\mathfrak{q}, x_1^w)}^M([\varphi]) \stackrel{\text{Claim 1}}{=} (\Psi_M \circ \Theta_{(\mathfrak{q}, x_1)}^M \circ \text{Ext}_S^d(\nu_w, \text{id}_M))([\varphi]) \neq 0.$$

In view of the definition of $\epsilon_{(\mathfrak{q}, x_1^w)}$, it is readily seen that

$$(4.3) \quad \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \overline{\varphi})(\epsilon_{(\mathfrak{q}, x_1^w)}) = [\varphi].$$

Therefore,

$$\begin{aligned} H_{(\mathfrak{q}, x_1^w)}^d(\overline{\varphi})(x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)}) &= x_1^{w-1}H_{(\mathfrak{q}, x_1^w)}^d(\overline{\varphi}) \circ \Theta_{(\mathfrak{q}, x_1^w)}^{\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))}(\epsilon_{(\mathfrak{q}, x_1^w)}) \\ &= x_1^{w-1}\Theta_{(\mathfrak{q}, x_1^w)}^M \circ \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \overline{\varphi})(\epsilon_{(\mathfrak{q}, x_1^w)}) \\ &= x_1^{w-1}\Theta_{(\mathfrak{q}, x_1^w)}^M([\varphi]), && \text{by Eq. (4.3)} \\ &\neq 0, && \text{by Eq. (4.2)} \end{aligned}$$

so Claim 2 follows.

One can readily verify that $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)} \neq 0$ in S if and only if $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)\widehat{S}} \neq 0$ in \widehat{S} , one approach would be using the faithful flatness of the completion map which induces an injective map on the relevant Ext-modules as well as considering each canonical element as an element of the direct limit of Ext-modules (see also [Ho83, Corollary (3.6) and Proposition (3.18)]). Similarly, the composited map in part (ii) is non-zero if and only if the analogue map $\text{Ext}_{\widehat{S}}^d(\widehat{S}/(\mathfrak{q}, x_1^w), \omega_{\widehat{S}}) \rightarrow \text{Ext}_{\widehat{S}}^d(\widehat{S}/(\mathfrak{q}, x_1), \omega_{\widehat{S}}) \rightarrow H_{(\mathfrak{q}, x_1)}^d(\omega_{\widehat{S}})$ is non-zero (we recall that $\omega_{\widehat{S}} = \omega_S \otimes_S \widehat{S}$). Thus in order to prove the equivalence of (i) and (ii), without loss of generality, we can assume that S is complete.

By [Ho83, page 534] (see also the proof of [Ho83, Theorem (4.3)]), there exists an S -homomorphism $\delta : H_{(\mathfrak{q}, x_1^w)}^d(\omega_S) \rightarrow E(S/\mathfrak{m}_S)$ which induces the functorial isomorphism

$$\rho_M : \text{Hom}_S(M, \omega_S) \rightarrow H_{(\mathfrak{q}, x_1^w)}^d(M)^\vee, \quad f \mapsto (h \mapsto \delta \circ H_{(\mathfrak{q}, x_1^w)}^d(f)(h)).$$

Having the isomorphism $\rho := \rho_{\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))}$, we can construct the diagram

$$\begin{array}{ccccccc} \text{Hom}_S(\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)), \omega_S) & \xrightarrow{\rho \cong} & H_{(\mathfrak{q}, x_1^w)}^d(\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)))^\vee & \xrightarrow{\gamma^\vee} & S^\vee & \xrightarrow{\Xi \cong} & E(S/\mathfrak{m}_S) \\ \downarrow \beta: f \mapsto \text{Ext}_S^d(\text{id}, f)(\epsilon_{(\mathfrak{q}, x_1^w)}) & & & & & & \uparrow \delta \\ \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) & \xrightarrow{\text{Ext}_S^d(\nu_w, \text{id})} & \text{Ext}_S^d(S/(\mathfrak{q}, x_1), \omega_S) & \xrightarrow{\Theta_{(\mathfrak{q}, x_1)}^{\omega_S}} & H_{(\mathfrak{q}, x_1)}^d(\omega_S) & \xrightarrow[\Psi]{\cong} & H_{(\mathfrak{q}, x_1^w)}^d(\omega_S) \end{array}$$

where $\gamma : S \rightarrow H_{(\mathfrak{q}, x_1^w)}^d(\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)))$ is given by $1 \rightarrow x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)}$, and $\Xi : S^\vee \rightarrow E(S/\mathfrak{m}_S)$ is the evaluation map at 1, thus $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)} \neq 0$ precisely when $\Xi \circ \gamma^\vee \circ \rho \neq 0$. Thus, if we can show that the diagram is commutative, then from the non-vanishing of $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)} \neq 0$ we get the non-vanishing of $\Theta_{(\mathfrak{q}, x_1)}^{\omega_S} \circ \text{Ext}_S^d(\nu_w, \text{id})$. The reverse implication also holds in view of Claim 2. Consequently, the proof is complete once the commutativity of the diagram is shown. This commutativity can be readily checked using Claim 1, so we leave it to the reader to check that the diagram is commutative.

(ii) \Leftrightarrow (iii): Let, $\pi_1 : S/(\mathfrak{q}, x_1^w) \rightarrow S/(\mathfrak{q}, x_1)$ be the natural epimorphism. We take into account the diagram

$$\begin{array}{ccccc} \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) & \xrightarrow{\text{Ext}_S^d(\nu_w, \text{id})} & \text{Ext}_S^d(S/(\mathfrak{q}, x_1), \omega_S) & \xrightarrow{\Theta_{(\mathfrak{q}, x_1)}^{\omega_S}} & H_{(\mathfrak{q}, x_1)}^d(\omega_S) \\ & \searrow x_1^{w-1} \cdot \text{id} & \downarrow \text{Ext}_S^d(\pi_1, \text{id}) & & \downarrow \Psi \\ & & \text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) & \xrightarrow{\Theta_{(\mathfrak{q}, x_1^w)}^{\omega_S}} & H_{(\mathfrak{q}, x_1^w)}^d(\omega_S) \end{array}$$

which is commutative in view of Claim 1 in the proof of the previous implications. Then the statement is clear in view of the commutativity of this diagram as well as the bijectivity of Ψ . \square

Consideration 4.5. Let (S, \mathfrak{m}_S) be a local ring of dimension d . Throughout the rest of this section we will repeatedly take into account the following data:

- a system of parameters $\mathbf{x} := x_1, \dots, x_d$ of S .
- some height $d - 1$ ideal \mathfrak{q} containing x_2, \dots, x_d (but not x_1).
- some $w \in \mathbb{N}$.
- a (not necessarily minimal) free resolution G_\bullet of $S/(\mathfrak{q}, x_1^w)$.

Proposition-Definition 4.6. (A variant of the Canonical Element Theorem) Let (S, \mathfrak{m}_S) be a local ring of dimension d . The following statements are equivalent.

(i) For any choice of $\mathbf{x}, \mathfrak{q}, w, G_\bullet$ as in Consideration 4.5, any chain map

$$\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d; S) \rightarrow G_\bullet \quad \text{s.t.} \quad \overline{\phi_0}(r + (x_1^w, x_2, \dots, x_d)) = r + (\mathfrak{q}, x_1^w)$$

and any $v \in \mathbb{N}$,

$$(x_1^{wv-w} x_2^{v-1} \dots x_d^{v-1}) \cdot (x_1^{w-1} \phi_d(1)) + \text{im}(\partial_{d+1}^{G_\bullet}) \notin (x_1^{wv}, x_2^v, \dots, x_d^v) \text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)).$$

(ii) Statement (i) holds when $v = 1$ is fixed, i.e.

$$x_1^{w-1} \phi_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \notin (x_1^w, x_2, \dots, x_d) \text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)).$$

(iii) For any $\mathbf{x}, \mathfrak{q}, w, G_\bullet$ as in Consideration 4.5, and for any chain map

$$\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d; S) \rightarrow G_\bullet \quad \text{s.t.} \quad \overline{\phi_0}(r + (x_1^w, x_2, \dots, x_d)) = x_1^{w-1} \cdot r + (\mathfrak{q}, x_1^w)$$

we have, $\phi_d \neq 0$.

(iv) For any $\mathbf{x}, \mathfrak{q}, w$ as in Consideration 4.5,

$$x_1^{w-1} \eta_{(\mathfrak{q}, x_1^w)} \neq 0.$$

(v) (Assuming that S admits a canonical module) For any $\mathbf{x}, \mathfrak{q}, w$ as in Consideration 4.5, the composited map

$$\text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) \xrightarrow{\text{Ext}_S^d(\nu_w, id_{\omega_S})} \text{Ext}_S^d(S/(\mathfrak{q}, x_1), \omega_S) \xrightarrow{\Theta_{(\mathfrak{q}, x_1)}^{\omega_S}} H_{(\mathfrak{q}, x_1)}^d(\omega_S)$$

is non-zero, where $\nu_w : S/(\mathfrak{q}, x_1) \rightarrow S/(\mathfrak{q}, x_1^w)$ is the multiplication map by x_1^{w-1} and $\Theta_{(\mathfrak{q}, x_1)}^{\omega_S}$ is the natural map to the local cohomology.

(vi) (Assuming that S admits a canonical module) For any $\mathbf{x}, \mathfrak{q}, w$ as in Consideration 4.5, the composited map

$$\text{Ext}_S^d(S/(\mathfrak{q}, x_1^w), \omega_S) \xrightarrow{\Theta_{(\mathfrak{q}, x_1^w)}^{\omega_S}} H_{(\mathfrak{q}, x_1^w)}^d(\omega_S) \xrightarrow{x_1^{w-1} id} H_{(\mathfrak{q}, x_1^w)}^d(\omega_S)$$

is non-zero.

(vii) The local ring S satisfies the Strong Canonical Element Property.

Proof. (i) \Leftrightarrow (ii): Only one side needs a proof. The first guess would be that this implication follows by the same argument as in [Ho83, Remark (2.2)(7)]. But some more work is required in this situation. Because if we change our under consideration system of parameters from x_1^w, x_2, \dots, x_d to $x_1^{wv}, x_2^v, \dots, x_d^v$, then we have to also consider $\text{syz}_d(S/(\mathfrak{q}, x_1^{wv}))$ in place of $\text{syz}_d(S/(\mathfrak{q}, x_1^w))$.

Arguing by contradiction, we consider a chain map ϕ_\bullet as in (i) such that

$$(4.4) \quad (x_1^{wv-w} x_2^{v-1} \dots x_d^{v-1}) \cdot (x_1^{w-1} \phi_d(1)) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^{wv}, x_2^v, \dots, x_d^v) \text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))$$

for some $v \geq 2$ (if $v = 1$ then we already get a contradiction). Without loss of generality, we can suppose that \mathfrak{q} is a prime ideal. Namely, since \mathfrak{q} has height $d - 1$ and \mathbf{x} is a system of parameters, so there exists

a minimal prime \mathfrak{p} of \mathfrak{q} such that $x_1 \notin \mathfrak{p}$. Then a composition of ϕ_\bullet with a chain map from the free resolution of $S/(\mathfrak{q}, x_1^w)$ to a free resolution of $S/(\mathfrak{p}, x_1^w)$ that lifts the natural epimorphism on the 0-th homologies yields a similar equation as in (4.4) but in the d -th syzygy of $S/(\mathfrak{p}, x_1^w)$. Henceforth, until the end of proof of this implication we assume that \mathfrak{q} is a prime ideal.

Taking into account the chain map ${}_v\eta_\bullet : K_\bullet(x_1^{wv}, x_2^v, \dots, x_d^v; S) \rightarrow K_\bullet(x_1^w, x_2, \dots, x_d; S)$ as in Lemma 2.2(ii) and setting

$$\psi_\bullet := \phi_\bullet \circ {}_v\eta_\bullet : K_\bullet(x_1^{wv}, x_2^v, \dots, x_d^v; S) \rightarrow G_\bullet$$

we get

$$(4.5) \quad x_1^{w-1}\psi_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^{wv}, x_2^v, \dots, x_d^v)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)).$$

Next, we consider the minimal free resolution ${}_wG_\bullet$ (respectively, ${}_{wv}G_\bullet$) of $S/(\mathfrak{q}, x_1^w)$ (respectively, $S/(\mathfrak{q}, x_1^{wv})$) as in Remark 4.2 as well as the chain maps

$$K_\bullet(x_1^{wv}, x_2^v, \dots, x_d^v; S) \xrightarrow{\gamma_\bullet} {}_{wv}G_\bullet \xrightarrow{{}_{wv,w}\lambda_\bullet} {}_wG_\bullet$$

where γ_\bullet is induced by the natural epimorphism on 0-th homologies and ${}_{wv,w}\lambda_\bullet$ is as in Lemma 4.3(i). Therefore, as ${}_{wv,w}\lambda_\bullet = \text{id}_S$, so from Lemma 2.1(ii) and display (4.5) we get

$$(4.6) \quad x_1^{w-1} \cdot ({}_{wv,w}\lambda_d \circ \gamma_d)(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^{wv}, x_2^v, \dots, x_d^v)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))$$

(because by mapping G_\bullet to ${}_wG_\bullet$ appropriately, we can presume that the target of ψ_\bullet is ${}_wG_\bullet$).

Next, in view of Lemma 4.3, we take into account ${}_{w,wv}\lambda_\bullet : {}_wG_\bullet \rightarrow {}_{wv}G_\bullet$ which has the property ${}_{w,wv}\lambda_\bullet \circ {}_{wv,w}\lambda_\bullet = x_1^{w(v-1)}\text{id}_{{}_{wv}G_\bullet}$, so from display (4.6) and considering ${}_{w,wv}\lambda_\bullet \circ ({}_{wv,w}\lambda_\bullet \circ \gamma_\bullet)$ in conjunction with ${}_{w,wv}\lambda_d \circ {}_{wv,w}\lambda_d = x_1^{w(v-1)}\text{id}_{{}_{wv}G_d}$ we get

$$(4.7) \quad x_1^{wv-1} \cdot \gamma_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^{wv}, x_2^v, \dots, x_d^v)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^{wv})).$$

This contradicts (ii).

(i) \Leftrightarrow (iv): This is immediate in light of [Ho83, Theorem (3.12)].

(ii) \Rightarrow (iii): Pick some chain map $\psi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d) \rightarrow G_\bullet$ lifting the natural epimorphism on the 0-th homologies. If there is another chain map $\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d) \rightarrow G_\bullet$ as in the statement of (iii) such that $\phi_d = 0$, then since $x_1^{w-1}\psi_\bullet$ and ϕ_\bullet both induce the same map on the 0-th homologies, so from Lemma 2.1(ii) and our hypothesis we get

$$x_1^{w-1}\psi_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^w, x_2, \dots, x_d)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))$$

which is a contradiction.

(iii) \Rightarrow (ii): Suppose, to the contrary that, for some chain map $\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d) \rightarrow G_\bullet$ as in the statement of (ii), we have

$$x_1^{w-1}\phi_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^w, x_2, \dots, x_d)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w)).$$

Setting $\psi_\bullet := x_1^{w-1}\phi_\bullet$, it is a chain map such that $\psi_0(r + (x_1^w, x_2, \dots, x_d)) = (rx_1^{w-1} + (\mathfrak{q}, x_1^w))$ on the 0-th homologies and such that $\psi_d(1) + \text{im}(\partial_{d+1}^{G_\bullet}) \in (x_1^w, x_2, \dots, x_d)\text{syz}_d^{G_\bullet}(S/(\mathfrak{q}, x_1^w))$, say

$$\psi_d(1) = x_1^w\alpha_1 + \sum_{i=2}^d \alpha_i x_i + \partial_{d+1}^{G_\bullet}(\beta), \quad \alpha_1, \dots, \alpha_d \in G_d, \beta \in G_{d+1}.$$

Then, we turn ψ_\bullet into another chain map $\psi'_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d) \rightarrow G_\bullet$ as follows. We set $\psi'_i = \psi_i$ for each $0 \leq i \leq d-2$ and $\psi'_d = 0$, while ψ'_{d-1} is defined as

$$\psi'_{d-1}(e_1 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_d) = \psi_{d-1}(e_1 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_d) - (-1)^{i+1} \partial_d^{G_\bullet}(\alpha_i).$$

Then ψ' satisfies all of the conditions of (iii), but the last non-vanishing condition of it. This is a contradiction.

(iv) \Leftrightarrow (v) \Leftrightarrow (vi): Without loss of generality, we can assume that R is complete thus it is a homomorphic image of a Gorenstein ring. Then, these implications hold in view of Proposition 4.4. \square

Remark 4.7. If in the statement of any part of the previous proposition we set $w = 1$, then it follows from the original Canonical Element Theorem.

Remark 4.8. At the time of writing this paper the author does not know any variant of the Improved New Intersection Theorem which is equivalent to the Strong Canonical Element Property.

The first part of the following corollary shows how $\eta_{(\mathfrak{q}, x_1)}$ and $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)}$ are connected. The second part of the corollary can perhaps be useful to connect the non-vanishing of $x_1^{w-1}\eta_{(\mathfrak{q}, x_1^w)}$ (for some w) to the formal local cohomology? The syzygies in the statement and the proof of the next corollary are computed with respect to the resolutions as in Remark 4.2 and Lemma 4.3.

Corollary 4.9. *Let $\mathbf{x}, \mathfrak{q}, w$ be as in Consideration 4.5 and assume that \mathfrak{q} is a prime ideal.*

(i) *There are S -homomorphisms*

$$H_{(\mathfrak{q}, x_1^w)}^d \left(\text{syz}_d(S/(\mathfrak{q}, x_1^w)) \right) \xrightarrow{\mu_{w,1}} H_{(\mathfrak{q}, x_1)}^d \left(\text{syz}_d(S/(\mathfrak{q}, x_1)) \right) \xrightarrow{\mu_{1,w}} H_{(\mathfrak{q}, x_1^w)}^d \left(\text{syz}_d(S/(\mathfrak{q}, x_1^w)) \right)$$

such that

$$\mu_{w,1}(\eta_{(\mathfrak{q}, x_1^w)}) = \eta_{(\mathfrak{q}, x_1)}, \quad \mu_{1,w} \circ \mu_{w,1} = x_1^{w-1} \text{id}.$$

In particular,

$$\mu_{1,w}(\eta_{(\mathfrak{q}, x_1)}) = x_1^{w-1} \eta_{(\mathfrak{q}, x_1^w)}.$$

(ii) *Suppose that x_1 is a regular element of S . Let*

$$\delta_{\mathbf{x}, \mathfrak{q}, w} : H_{(\mathfrak{q}, x_1)}^{d-1} \left(\text{syz}_d(S/\mathfrak{q})/x_1^{w-1} \text{syz}_d(S/\mathfrak{q}) \right) \rightarrow H_{(\mathfrak{q}, x_1)}^d \left(\text{syz}_d(S/(\mathfrak{q}, x_1)) \right)$$

be the connecting homomorphism induced by the exact sequence in Lemma 4.3(iii). A necessary and sufficient condition for the validity of Proposition-Definition 4.6(iv) for this choice of $\mathbf{x}, \mathfrak{q}, w$, is that

$$\eta_{(\mathfrak{q}, x_1)} \notin \text{im}(\delta_{\mathbf{x}, \mathfrak{q}, w}).$$

Proof. Let $\zeta : S/(\mathfrak{q}, x_1^w) \rightarrow S/(\mathfrak{q}, x_1)$ be the natural epimorphism.

(i) Following the notation of Lemma 4.3 we consider the commutative diagram,

$$\begin{array}{ccccc} \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1^w)}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1^w)} \right) \right) & \xrightarrow{\text{Ext}_S^d(\text{id}, f_{w,1})} & \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1^w)}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1)} \right) \right) & \xleftarrow{\text{Ext}_S^d(\zeta, \text{id})} & \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1)}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1)} \right) \right) \\ \Theta_{(\mathfrak{q}, x_1^w)}^{\text{syz}_d(S/(\mathfrak{q}, x_1^w))} \downarrow & & \Theta_{(\mathfrak{q}, x_1^w)}^{\text{syz}_d(S/(\mathfrak{q}, x_1))} \downarrow & & \Theta_{(\mathfrak{q}, x_1)}^{\text{syz}_d(S/(\mathfrak{q}, x_1))} \downarrow \\ \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1^w)^m}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1^w)} \right) \right) & \xrightarrow{H^*} & \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1^w)^m}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1)} \right) \right) & \xleftarrow{\Psi_1} & \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d \left(\frac{S}{(\mathfrak{q}, x_1)^m}, \text{syz}_d \left(\frac{S}{(\mathfrak{q}, x_1)} \right) \right) \end{array}$$

where $H^* = H_{(\mathfrak{q}, x_1^w)}^d(f_{w,1})$ and it is well-known that the natural induced map Ψ_1 on the limit of Ext modules is an isomorphism. We define

$$\mu_{w,1} := \Psi_1^{-1} \circ H_{(\mathfrak{q}, x_1^w)}^d(f_{w,1}).$$

In view of the definition of the element ϵ_I assigned to ideal I and the induced maps on Ext-modules, it is readily verified that $\text{Ext}_S^d(\zeta, \text{id})(\epsilon_{(\mathfrak{q}, x_1)}) = \text{Ext}_S^d(\text{id}, f_{w,1})(\epsilon_{(\mathfrak{q}, x_1^w)})$, so the above commutative diagram shows that $\mu_{w,1}(\eta_{(\mathfrak{q}, x_1^w)}) = \eta_{(\mathfrak{q}, x_1)}$. Considering the induced map

$$\Psi_w : \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1)^m, \text{syz}_d(S/(\mathfrak{q}, x_1^w))\right) \rightarrow \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1^w)^m, \text{syz}_d(S/(\mathfrak{q}, x_1^w))\right)$$

by ζ on direct limits of Ext modules, we set

$$\mu_{1,w} := \Psi_w \circ H_{(\mathfrak{q}, x_1)}^d(f_{1,w}) : \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1)^m, \text{syz}_d(S/(\mathfrak{q}, x_1))\right) \rightarrow \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1^w)^m, \text{syz}_d(S/(\mathfrak{q}, x_1^w))\right).$$

Finally from the commutative diagram,

$$\begin{array}{ccc} \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1)^m, \text{syz}_d(S/(\mathfrak{q}, x_1))\right) & \xrightarrow{\Psi_1} & \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1^w)^m, \text{syz}_d(S/(\mathfrak{q}, x_1))\right) \\ \downarrow H_{(\mathfrak{q}, x_1)}^d(f_{1,w}) & & \downarrow H_{(\mathfrak{q}, x_1^w)}^d(f_{1,w}) \\ \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1)^m, \text{syz}_d(S/(\mathfrak{q}, x_1^w))\right) & \xrightarrow{\Psi_w} & \lim_{\substack{\longrightarrow \\ m \in \mathbb{N}}} \text{Ext}_S^d\left(S/(\mathfrak{q}, x_1^w)^m, \text{syz}_d(S/(\mathfrak{q}, x_1^w))\right) \end{array}$$

we get

$$\begin{aligned} \mu_{1,w} \circ \mu_{w,1} &= \Psi_w \circ H_{(\mathfrak{q}, x_1)}^d(f_{1,w}) \circ \Psi_1^{-1} \circ H_{(\mathfrak{q}, x_1^w)}^d(f_{w,1}) = H_{(\mathfrak{q}, x_1^w)}^d(f_{1,w}) \circ \Psi_1 \circ \Psi_1^{-1} \circ H_{(\mathfrak{q}, x_1^w)}^d(f_{w,1}) \\ &= H_{(\mathfrak{q}, x_1^w)}^d(f_{1,w} \circ f_{w,1}) \\ &= H_{(\mathfrak{q}, x_1^w)}^d\left(x_1^{w-1} \text{id}_{\text{syz}_d(S/(\mathfrak{q}, x_1^w))}\right) \quad (\text{by Lemma 4.3}) \\ &= x_1^{w-1} \text{id}_{H_{(\mathfrak{q}, x_1^w)}^d(\text{syz}_d(S/(\mathfrak{q}, x_1^w)))}. \end{aligned}$$

(ii) By the previous part, $x_1^{w-1} \eta_{(\mathfrak{q}, x_1^w)} \neq 0$ if and only if $\mu_{1,w}(\eta_{(\mathfrak{q}, x_1)}) \neq 0$ and by definition of $\mu_{1,w}$ this holds precisely when $H_{(\mathfrak{q}, x_1)}^d(f_{1,w})(\eta_{(\mathfrak{q}, x_1)}) \neq 0$ (as Ψ_w is an isomorphism). Thus, the long exact sequence obtained from the exact sequence of Lemma 4.3(iii) and $\Gamma_{(\mathfrak{q}, x_1)}(-)$ yields the statement. \square

4.2. The validity of our variant of the Canonical Element Theorem. In the remainder of this section, we first show that if S has a balanced big Cohen-Macaulay module satisfying two additional properties, then S has the Strong Canonical Element Property (Lemma 4.15). Thereafter, we will show that a balanced big Cohen-Macaulay module with our desired additional properties exists over any normal excellent local domain and so demonstrate that the Strong Canonical Element Property holds in general (Corollary 4.19). We begin with a preparatory remark, definition, convention and lemma.

The following remark will be referred to later.

Remark 4.10. Suppose that $\varphi : (S, \mathfrak{m}_S) \rightarrow (S', \mathfrak{m}_{S'})$ is a local ring homomorphism such that the image, $\varphi(\mathbf{x})$, of a system of parameters $\mathbf{x} := x_1, \dots, x_d$ of S forms a system of parameters for S' . Let \mathfrak{q} (respectively, \mathfrak{q}') be a height $d-1$ ideal of S (respectively, of S') such that $x_2, \dots, x_d \in \mathfrak{q}$ (respectively, $\varphi(x_2), \dots, \varphi(x_d) \in \mathfrak{q}'$). Assume that $\mathfrak{q}S' \subseteq \mathfrak{q}'$. If the conclusion of any of propositions 4.6(i), (ii) or (iii) holds in S' for this $\varphi(\mathbf{x})$, \mathfrak{q}' , w and any chain map as in the statement, then it also holds for S for \mathbf{x} , \mathfrak{q} , w and any chain map as in the statement. Because, if otherwise, then by tensoring a counterexample into S' and then mapping $G_\bullet \otimes_S S'$ to a free resolution of $S'/(\mathfrak{q}', \varphi(x_1^w))$ we get a contradiction in S' for the triple $\varphi(\mathbf{x})$, \mathfrak{q}' , w .

Definition 4.11. ([BrHe98, Definition 9.1.1]) Let S be a ring (possibly non-Noetherian), I an ideal generated by $\mathbf{x} := x_1, \dots, x_n$ and M be an R -module. If all Koszul homologies $H_i(\mathbf{x}, M)$ vanish, then we set $\text{grade}(I, M) = \infty$; otherwise, $\text{grade}(I, M) = n - v$ where $v = \sup\{i : H_i(\mathbf{x}, M) \neq 0\}$.

Convention 4.12. In what follows when no specific generating set for I is determined, by abuse of notation, we let $H_i(I, M)$ denote the i -th Koszul homology of M with respect to any generating set of I . For our purpose here, which is the realization of grade in terms of Koszul homologies, this convention causes no confusion (see the paragraph after [BrHe98, Definition 9.1.1]).

See Remark 4.14 for some comments on the imposed conditions in the statement of Lemma 4.13(ii).

Lemma 4.13. *Let (S, \mathfrak{m}_S) be a local ring, \mathfrak{a} be an ideal of S generated by n elements and M be an S -module.*

- (i) $H_{\mathfrak{a}}^i(M) = 0$ for each $0 \leq i < \text{grade}(\mathfrak{a}, M)$.
- (ii) Suppose that M is a balanced big Cohen-Macaulay S -module and that $\text{grade}(\mathfrak{a}, M) = g = \text{height}(\mathfrak{a})$. Then the map

$$H_{n-g}(\mathfrak{a}, M) \rightarrow H_{\mathfrak{a}}^g(M)$$

(as in [BrSh13, Theorem 5.2.9]) is injective. In particular,

$$H_{\mathfrak{a}}^g(M) \neq 0.$$

Proof. (i) First from [BrHe98, Proposition 9.1.3(b)] we get $\text{grade}(\mathfrak{a}^s, M) = \text{grade}(\mathfrak{a}, M)$ for each $s \in \mathbb{N}$, therefore $\text{Ext}_S^i(S/\mathfrak{a}^s, M) = 0$ for each $0 \leq i < \text{grade}(\mathfrak{a}, M)$ and each $s \in \mathbb{N}$ by [BrHe98, Exercise 9.1.10(d)]. Consequently,

$$\forall 0 \leq i < \text{grade}(\mathfrak{a}, M), \quad H_{\mathfrak{a}}^i(M) = 0 \quad ([\text{BrSh13, Theorem 1.3.8}]).$$

(ii) We use induction on g . If $g = 0$ then this map is the inclusion $0 :_M \mathfrak{a} \rightarrow \Gamma_{\mathfrak{a}}(M)$ and the statement holds. Suppose that $g > 0$ and the statement holds for $g - 1$.

Since $\text{height}(\mathfrak{a}) > 0$, so \mathfrak{a} contains some $x \in \mathfrak{a}$ such that $x \in S^\circ = S \setminus \cup_{\mathfrak{p} \in \min(S)} \mathfrak{p}$, in particular x forms a parameter element in S and in any localization $S_{\mathfrak{p}'}$ of S . Thus

$$(4.8) \quad \text{height}_{S/xS}(\mathfrak{a}/xS) = \text{height}(\mathfrak{a}) - 1 = g - 1.$$

Moreover, since M is assumed to be a balanced big Cohen-Macaulay module, so x is regular over M , M/xM is a balanced big Cohen-Macaulay S/xS -module ([Sh81, Lemma (2.3)]) and finally

$$(4.9) \quad \text{grade}(\mathfrak{a}/xS, M/xM) \stackrel{[\text{BrHe98, Proposition 9.1.2(b)}]}{=} g - 1 \stackrel{\text{Eq. (4.8)}}{=} \text{height}_{S/xS}(\mathfrak{a}/xS).$$

Using long exact sequences arising from $0 \rightarrow M \rightarrow M \rightarrow M/xM \rightarrow 0$, we get the diagram

$$\begin{array}{ccccccc} \underbrace{H_{\mathfrak{a}}^{g-1}(M)}_{\substack{=0 \\ \text{Part (i)}}} & \longrightarrow & H_{\mathfrak{a}}^{g-1}(M/xM) & \xrightarrow{\text{injective}} & H_{\mathfrak{a}}^g(M) & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \underbrace{H_{n-(g-1)}(\mathfrak{a}, M)}_{\substack{=0 \\ \text{(definition of grade)}}} & \longrightarrow & H_{n-(g-1)}(\mathfrak{a}, M/xM) & \xrightarrow{\cong} & H_{n-g}(\mathfrak{a}, M) & \longrightarrow & \underbrace{xH_{n-g}(\mathfrak{a}, M)}_{\substack{=0 \\ ([\text{BrHe98, Proposition 1.6.5(b)}])}} \end{array}$$

Hence, it suffices to notice that $H_{n-(g-1)}(\mathfrak{a}, M/xM) \rightarrow H_{\mathfrak{a}}^{g-1}(M/xM)$ is injective, which holds true by our inductive hypothesis together with display (4.9) and the balanced big Cohen-Macaulayness of M/xM (over S/xS) as well as the fact that $H_{n-(g-1)}(\mathfrak{a}, M/xM)$ and $H_{\mathfrak{a}}^{g-1}(M/xM)$ both can be realized as the same corresponding Koszul homology and local cohomology over the ring S/xS (with respect to $\mathfrak{a}(S/xS)$). \square

Remark 4.14. In the proof of the second part of the previous lemma the assumption on the balanced big Cohen-Macaulayness of M is used (thus perhaps is necessary) for deducing the existence of an M -regular element $x \in \mathfrak{a}$ from $\text{grade}(\mathfrak{a}, M) > 0$ (see also [Sh81, Theorem (2.6)] and [BrHe98, Exercise 9.1.10(d)]). Then the assumption on $\text{grade}(\mathfrak{a}, M) = \text{height}(\mathfrak{a})$ is imposed to enable us to pick $x \in \mathfrak{a}$ so that M/xM is a balanced big Cohen-Macaulay S/xS -module as well, which is needed for our inductive step (cf. Remark 4.16).

Lemma 4.15. *Fix some $\mathfrak{p}, \mathbf{x}, w, G_\bullet$ as in Consideration 4.5 such that \mathfrak{p} is a prime ideal³. Suppose that there is a balanced big Cohen-Macaulay S -module M such that*

- (i) $\mathfrak{p} \in \text{Ass}_S(M/(x_2, \dots, x_d)M)$, or equivalently $\text{grade}(\mathfrak{p}, M) = d - 1$. In the case where S is (furthermore) a catenary domain, this is also equivalent to say that $M_{\mathfrak{p}}$ is a balanced big Cohen-Macaulay $S_{\mathfrak{p}}$ -module.
- (ii) $M/(x_2, \dots, x_d)M$ is \mathfrak{m}_S -adically separated.

Then Proposition-Definition 4.6(ii) holds for $\mathfrak{p}, \mathbf{x}, w, G_\bullet$ and any chain map $\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d; S) \rightarrow G_\bullet$ lifting the natural epimorphism on 0-th homologies.

Proof. First we notice that, if $\mathfrak{p} \in \text{Ass}_S(M/(x_2, \dots, x_d)M)$ then $\text{grade}(\mathfrak{p}, M/(x_2, \dots, x_d)M) = 0$ ([BrHe98, Proposition 9.1.2(a)], i.e. $\text{grade}(\mathfrak{p}, M) = d - 1$ ([BrHe98, Proposition 9.1.2(b)]). Conversely, suppose that $\text{grade}(\mathfrak{p}, M) = d - 1$ or equivalently $\text{grade}(\mathfrak{p}, M/(x_2, \dots, x_d)M) = 0$. Then $M/(x_2, \dots, x_d)M$ is a balanced big Cohen-Macaulay module over $S/(x_2, \dots, x_d)$ ([Sh81, Lemma (2.3)]), therefore we must have $\mathfrak{p} \in \text{Ass}(M/(x_2, \dots, x_d)M)$ because \mathfrak{p} annihilates a non-zero element of $M/(x_2, \dots, x_d)M$ in view of [BrHe98, Proposition 9.1.2(a)], while $\mathfrak{m} \notin \text{Ass}(M/(x_2, \dots, x_d)M)$ by [BrHe98, Proposition 8.5.5]. If S is a catenary domain, for the mentioned equivalence in (i) with the balanced big Cohen-Macaulayness of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$ we refer to [Sh82, Theorem 4.3]. Here we remark further that if $M_{\mathfrak{p}}$ is a balanced big Cohen-Macaulay $S_{\mathfrak{p}}$ -module, then $\text{grade}(\mathfrak{p}S_{\mathfrak{p}}, M_{\mathfrak{p}}) = \text{height}(\mathfrak{p})$ by [BrHe98, Exercise 9.1.12]⁴ from which we can deduce that $\mathfrak{p} \in \text{Ass}(M/(x_2, \dots, x_d)M)$ as above.

Suppose to the contrary that there is some chain map $\phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d; S) \rightarrow G_\bullet$ lifting the natural epimorphism on 0-th homologies, for which Proposition-Definition 4.6(ii) does not hold. Let $e := \phi_0(1)$ be a basis of G_0 representing the homology class $[1 + (x_1^w, \mathfrak{p})]$ in $H_0(G_\bullet)$, and let $\mathbf{x}' := x_2, \dots, x_d$. By condition (i), there is some $b + \mathbf{x}'M \in M/\mathbf{x}'M$ with $\mathfrak{p} = (\mathbf{x}'M) :_S b \subseteq \mathfrak{m}_S$. Additionally, without loss of generality, we may and we do assume that

$$(4.10) \quad b \notin (\mathbf{x})M.$$

Namely, let $b \in \mathbf{x}M$. Then by condition (ii), $b \in (x_1^n, x_2, \dots, x_d)M \setminus (x_1^{n+1}, x_2, \dots, x_d)M$ for some $n \in \mathbb{N}$. Assuming $b = x_1^n b_1 + \sum_{i=2}^d x_i b_i$, we get $\mathfrak{p} = (\mathbf{x}'M) :_S x_1^n b_1$. From this and the balanced big Cohen-Macaulayness of M we get $\mathfrak{p} = (\mathbf{x}'M) :_S b_1$. On the other hand $b_1 \notin (\mathbf{x})M$ otherwise we get $b \in (x_1^{n+1}, x_2, \dots, x_d)M$ contradicting our choice of n .

Since $\mathfrak{p} \subseteq (\mathbf{x}'M :_S b)$ and $K_\bullet(x_1^w, x_2, \dots, x_d; M)$ is acyclic so there is a chain map

$$\theta_\bullet : G_\bullet \rightarrow K_\bullet(x_1^w, x_2, \dots, x_d; M)$$

such that $\theta_0(e) = b$, and consequently

$$\Delta_\bullet := \theta_\bullet \circ \phi_\bullet : K_\bullet(x_1^w, x_2, \dots, x_d; S) \rightarrow K_\bullet(x_1^w, x_2, \dots, x_d; M)$$

³Here we used the notation \mathfrak{p} in place of \mathfrak{q} when we referred to Consideration 4.5 because it is a prime ideal.

⁴See [BrHe98, page 350], for the notion of depth used in the cited exercise.

is a chain map such that $\Delta_0(1) = b$ while

$$(4.11) \quad x_1^{w-1}\Delta_d(1) \in (x_1^w, x_2, \dots, x_d)M.$$

In view of Lemma 2.1(i), Δ_\bullet is homotopic to

$$\Psi_\bullet : K_\bullet(x_1^w, \dots, x_d; S) \xrightarrow{\cong} K_\bullet(x_1^w, \dots, x_d; S) \otimes_S S \xrightarrow{\text{id} \otimes (1 \mapsto b)} K_\bullet(x_1^w, \dots, x_d; S) \otimes_S M \xrightarrow{\cong} K_\bullet(x_1^w, x_2, \dots, x_d; M)$$

with $\Psi_d(1) = b$ and $\Psi_0(1) = b$ and we get

$$\Delta_d(1) - \Psi_d(1) \in (x_1^w, x_2, \dots, x_d)M$$

(Lemma 2.1(ii)) and consequently

$$x_1^{w-1}\Delta_d(1) - x_1^{w-1}\Psi_d(1) \in (x_1^w, x_2, \dots, x_d)M.$$

From this and display (4.11) we get

$$x_1^{w-1}b = x_1^{w-1}\Psi_d(1) \in (x_1^w, x_2, \dots, x_d)M,$$

implying that $b \in (x_1, \dots, x_d)M$ as M is Cohen-Macaulay. This contradicts display (4.10). \square

Remark 4.16. Especially when S is a local ring with non-trivial zero divisors, it is possible to have a maximal Cohen-Macaulay S -module M and a prime ideal \mathfrak{p} of S such that $\text{grade}(\mathfrak{p}, M) \geq \text{height}(\mathfrak{p})$. For instance, let \mathfrak{m} be the maximal ideal of a regular local ring A of positive dimension and consider the Rees algebra $A[\mathfrak{m}t]$ and let S to be the localization of $A[\mathfrak{m}t]/(x^2t)$ at $(\mathfrak{m}, \mathfrak{m}t)$ where $x \in \mathfrak{m}$ is a non-zero element. Then $A = S/(\mathfrak{m}t)$ is a maximal Cohen-Macaulay S -algebra, $\text{height}(\mathfrak{m}S) = \text{height}(\mathfrak{m}A[\mathfrak{m}t]) - 1 = 0$, while $\text{grade}(\mathfrak{m}, S/(\mathfrak{m}t)) = \text{grade}(\mathfrak{m}, A) = \dim(A)$. Thus, when we are applying Lemma 4.15 even for rings admitting a maximal Cohen-Macaulay S -module M , we should be careful to be assured that the condition $\text{grade}(\mathfrak{p}, M) = d - 1 (= \text{height}(\mathfrak{p}))$ is satisfied. However if S is a normal excellent domain, then for any balanced big Cohen-Macaulay module M and any prime ideal \mathfrak{p} of S we have $\text{grade}(\mathfrak{p}, M) = \text{height}(\mathfrak{p})$ by Theorem 4.17. Thus in case S is an excellent normal domain and M is a balanced big Cohen-Macaulay S -module, we can apply Lemma 4.15 without checking condition (i).

Theorem 4.17. (cf. [BMPSTWW21, Proposition 2.11]) *Let (S, \mathfrak{m}_S) be an excellent local normal domain. Let M be a balanced big Cohen-Macaulay S -module. Then for any $\mathfrak{p} \in \text{Spec}(S)$, $M_{\mathfrak{p}}$ is a balanced big Cohen-Macaulay $S_{\mathfrak{p}}$ -module and $\text{grade}(\mathfrak{p}, M) = \text{height}(\mathfrak{p})$.*

Proof. Following [BrHe98, Exercise 8.5.8], we define the *small support* of M as

$$\text{supp}(M) := \{\mathfrak{p} \in \text{Spec}(S) : \mathfrak{p}M_{\mathfrak{p}} \neq M_{\mathfrak{p}}\}$$

while $\text{Supp}(M)$ denotes the usual notion of support that is $\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(S) : M_{\mathfrak{p}} \neq 0\}$.

First, we notice that $\text{Supp}(M) = \text{Spec}(S)$, as S is a domain and M is a balanced big Cohen-Macaulay module. Pick some $\mathfrak{p} \in \text{Spec}(S) \setminus \{\mathfrak{m}_S\}$ (the statement holds for \mathfrak{m}_S). In order to prove that $M_{\mathfrak{p}}$ is a balanced big Cohen-Macaulay $R_{\mathfrak{p}}$ -module and also to prove that $\text{grade}(\mathfrak{p}, M) = \text{height}(\mathfrak{p})$, by localization at a prime ideal \mathfrak{q} containing \mathfrak{p} with $\text{height}(\mathfrak{q}) = \text{height}(\mathfrak{p}) + 1$, we can see that it suffices to prove the statement under the assumption that $\text{height}(\mathfrak{p}) = \dim(S) - 1$. Namely, for the equality $\text{grade}(\mathfrak{p}, M) = \text{height}(\mathfrak{p})$ we notice that $\text{height}(\mathfrak{p}) \leq \text{grade}(\mathfrak{p}, M)$ as M is a balanced big Cohen-Macaulay ([BrHe98, Proposition 9.1.2(b)]) and thus

$$\begin{aligned} \text{height}(\mathfrak{p}) &\leq \text{grade}(\mathfrak{p}, M) \leq \text{grade}(\mathfrak{p}S_{\mathfrak{q}}, M_{\mathfrak{q}}), && \text{(definition of grade)} \\ &= \text{height}(\mathfrak{p}S_{\mathfrak{q}}), && \text{(statement holds over } S_{\mathfrak{q}}(\text{height}(\mathfrak{q}) = \text{height}(\mathfrak{p}) + 1)) \\ &= \text{height}(\mathfrak{p}). \end{aligned}$$

Hence, without loss of generality, we may and we do assume that $\text{height}(\mathfrak{p}) = \dim(S) - 1$. Since S is an excellent normal domain, thus so is its completion \widehat{S} , therefore we can appeal to [BrSh13, 8.2.1 The Lichtenbaum–Hartshorne Vanishing Theorem] to observe that $H_{\mathfrak{p}}^{\dim(S)}(S) = 0$ and thence $H_{\mathfrak{p}}^{\dim(S)}(M) = 0$, therefore we know that

$$(4.12) \quad \forall i \geq \dim(S), \quad H_{\mathfrak{p}}^i(M) = 0$$

by Grothendieck’s Vanishing Theorem.

Since $H_0(\mathfrak{p}, M) = M/\mathfrak{p}M \neq 0$, so $\text{grade}(\mathfrak{p}, M) < \infty$ by definition of grade (Definition 4.11). We show that $\text{grade}(\mathfrak{p}, M) = \text{height}(\mathfrak{p})$. If $\text{grade}(\mathfrak{p}, M) \geq \dim(S)$, then Lemma 4.13(i) in conjunction with display (4.12) implies that $H_{\mathfrak{p}}^i(M) = 0$ for all i . Then appealing to [Sc98, Corollary 1.4] for some (or any) $x \in \mathfrak{m}_S \setminus \mathfrak{p}$, from $H_{\mathfrak{p}}^*(M) = 0$ we conclude that $H_{\mathfrak{m}_S}^i(M) = 0$ for all i (we recall that $\text{height}(\mathfrak{p}) = \dim(S) - 1$) which is a contradiction ([BrHe98, Exercise 9.1.12(a) and (c)]). Therefore, $\text{grade}(\mathfrak{p}, M) \leq \dim(S) - 1$. On the other hand $\text{grade}(\mathfrak{p}, M) \geq \dim(S) - 1$ in view of [BrHe98, Proposition 9.1.2(b)], because \mathfrak{p} contains a part of a system of parameters of length $\dim(S) - 1$ which has to be a regular sequence on M . So $\text{grade}(\mathfrak{p}, M) = \dim(S) - 1 = \text{height}(\mathfrak{p})$ and thence Lemma 4.13(ii) and (i) implies that $H_{\mathfrak{p}}^{\dim(S)-1}(M)$ is the only non-zero local cohomology of M supported at \mathfrak{p} .

Let $h \in S \setminus \mathfrak{p}$. In view of [Sc98, Corollary 1.4], $H_{hS}^0(H_{\mathfrak{p}}^{\dim(S)-1}(M))$ is a homomorphic image of $H_{\mathfrak{m}_S}^{\dim(S)-1}(M)$ while the latter module is zero because of the big Cohen-Macaulayness of M . Therefore $H_{hS}^0(H_{\mathfrak{p}}^{\dim(S)-1}(M)) = 0$, i.e. the non-zero module $H_{\mathfrak{p}}^{\dim(S)-1}(M)$ (non-zero because of the conclusion of the previous paragraph) has no non-zero h -torsion element and thence $H_{\mathfrak{p}S_h}^{\dim(S)-1}(M_h) \neq 0$. From this fact together the maximality of $\mathfrak{p}S_h \in \text{Spec}(S_h)$ we get $\text{Supp}(H_{\mathfrak{p}S_h}^{\dim(S)-1}(M_h)) = \{\mathfrak{p}S_h\}$ and consequently $H_{\mathfrak{p}S_{\mathfrak{p}}}^{\dim(S)-1}(M_{\mathfrak{p}}) \neq 0$. Then the statement follows from [BrHe98, Exercise 8.5.9] and [Sh82, Theorem 4.3]. \square

The condition (ii) in Lemma 4.15 is available in view of the following remark.

Remark 4.18. Let (S, \mathfrak{m}_S) be a local ring and M be a big Cohen-Macaulay module (with respect to any system of parameters).

- (i) The \mathfrak{m}_S -adic completion \widehat{M} of M is a balanced big Cohen-Macaulay module ([BrHe98, Corollary 8.5.3]).
- (ii) If M is a balanced big Cohen-Macaulay module, then for each part of a system of parameters \mathbf{x}' of S we have the isomorphism $\widehat{M}/\mathbf{x}'\widehat{M} \cong \widehat{M/\mathbf{x}'M}$ by [St90, Theorem 5.2.3], so $\widehat{M}/\mathbf{x}'\widehat{M}$ is \mathfrak{m}_S -adically complete (see also [St90, Theorem 5.1.7] and [St90, Definition 5.1.13]).

Now we show all local rings satisfy the Strong Canonical Element Property. To this aim, we apply the existence of balanced big Cohen-Macaulay algebras thanks to [HoHu92], [HoHu95] and [An20]. In particular, the next corollary in this section and its characteristic dependent proof (induced by the characteristic dependent proof of the existence of balanced big Cohen-Macaulay algebras) have no relation with the content of the next section. So the characteristic free nature of the results of other sections, including the main results of our paper given in the next section, are unaffected.

Corollary 4.19. *Let (S, \mathfrak{m}_S) be a local ring. Then S satisfies the Strong Canonical Element Property.*

Proof. Let $d := \dim(S)$ and pick some $\mathfrak{q}, \mathbf{x}, w, G_{\bullet}$ as in Consideration 4.5 as well as some chain map $\phi_{\bullet} : K_{\bullet}(x_1^w, x_2, \dots, x_d; S) \rightarrow G_{\bullet}$ lifting the natural surjection on 0-th homologies. We show that Proposition-Definition 4.6(ii) holds for $\mathfrak{q}, \mathbf{x}, w, G_{\bullet}$ and ϕ_{\bullet} .

By Remark 4.10, it suffices to prove the statement for the case where S is complete. Also, without loss of generality we may and we do assume that \mathfrak{q} is prime. Since $\text{height}(\mathfrak{q}) = d - 1$, so we automatically have

$\text{height}(\mathfrak{q}) + \dim(S/\mathfrak{q}) = d$ and thus there exists a minimal prime ideal \mathfrak{p}_0 of S contained in \mathfrak{q} such that $\dim(S/\mathfrak{p}_0) = \dim(S)$. Therefore, in view of Remark 4.10, by passing to S/\mathfrak{p}_0 , without loss of generality we can assume that S is a domain. Then, similarly, by passing to the integral closure \overline{S} of S in $\text{Frac}(S)$ and then replacing \mathfrak{q} by a prime ideal of \overline{S} lying over \mathfrak{q} , without loss of generality, we can assume that S is a complete local normal domain.

Let B be an \mathfrak{m}_S -adically complete balanced big-Cohen-Macaulay S -algebra which exists in view of Remark 4.18(i) as well as [HoHu92], [HoHu95] and [An20] (see also [SP, Tag 05GG(1)]). Then by Theorem 4.17, or [BMPSTWW21, Proposition 2.11], we observe that $B_{\mathfrak{q}}$ is a balanced big Cohen-Macaulay $S_{\mathfrak{q}}$ -algebra and $\text{grade}(\mathfrak{q}, B) = d - 1$. Moreover, $B/(x_2, \dots, x_d)B$ is \mathfrak{m}_S -adically separated by Remark 4.18(ii). Therefore the statement follows from Lemma 4.15. \square

5. THE MAIN RESULTS

Applying the results of the previous sections, in this section we show that a possible deduction of our variant of the Canonical Element Theorem from the original Canonical Element Theorem provides us with a characteristic free proof of the Canonical Element Theorem. The same conclusion will be shown under an alternative assumption on the stability of the CE Property with respect to the localization over excellent factorial domains. Another result of this section is Corollary 5.2 which suggests an approach for settling the Balanced Big Cohen-Macaulay Module Conjecture by a characteristic free proof.

Theorem 5.1. *Let (R, \mathfrak{m}_R) be a local ring. Then R satisfies Hochster's Canonical Element Conjecture (followed by a characteristic free proof) provided either of the following assertions holds (in a characteristic free way):*

- (i) *For a fixed excellent factorial local domain that is a homomorphic image of a regular local ring, equivalent parts of Proposition-Definition 4.6 hold provided the original Canonical Element Conjecture holds.*
- (ii) *If a fixed excellent factorial local domain that is a homomorphic image of a regular local ring satisfies the Canonical Element Conjecture, then so does the localization of it at any prime ideal.*

Proof. We prove that R satisfies the CE property provided either of the assertions (i) or (ii) holds. We argue by induction on $n := \dim(R)$.

Since the statement is immediate in dimension ≤ 2 , so we assume that $n \geq 3$ and the statement has been proved for dimension less than n .

Step 1: Reduction to complete quasi-Gorenstein domains that are locally complete intersection in codimension ≤ 1 . By [Ho83, (3.9) Remark], we can reduce to the case where R is a complete local normal domain. Since R is a complete local domain, R admits a canonical module ω_R which is an ideal of R ([AoGo85, (3.1)]). Then we proceed as follows:

- (i) Using [Ta17, Lemma 2.2], we choose $a \in \mathfrak{m}_R$ which has no square root in $\text{Frac}(R)$. Then the R -algebra $R(a^{1/2}) := R[X]/(X^2 - a)$ is an integral domain (see [Ta17, Remark 2.1]). The element $a^{1/2} := X + (X^2 - a)$ is a square root of a in $R(a^{1/2})$.
- (ii) Here we follow [Ta17, Remark 2.3]. Let \mathfrak{k} be the kernel of the R -algebra homomorphism $R[\omega_R X] \hookrightarrow R[X] \twoheadrightarrow R[X]/(X^2 - a)$, where $R[\omega_R X]$ is the Rees algebra of the ideal ω_R . Then $R[\omega_R X]/\mathfrak{k}$ is a domain by part (i), as it is a subring of $R(a^{1/2})$. Since $X + (X^2 - a) = a^{1/2}$ in $R(a^{1/2})$, we denote $\omega_R X + \mathfrak{k}$ (respectively $r + wX + \mathfrak{k}$, where $r \in R$, $w \in \omega_R$) by $\omega_R a^{1/2}$ (respectively $r + wa^{1/2}$).
- (iii) We endow $R \oplus \omega_R \subseteq R \oplus R$ with an R -algebra structure by which it is a subalgebra of $R(a^{1/2})$ as R modules $\cong R \oplus R$. It is easily seen that the R -module homomorphism

$$R \oplus \omega_R \rightarrow R[\omega_R X]/\mathfrak{k}, \quad (r, w) \in R \oplus \omega_R \mapsto r + wa^{1/2}$$

is an isomorphism of R -modules inducing an R -algebra structure on $R \oplus \omega_R$ via

$$(r, w)(r', w') := (rr' + ww'a, rw' + r'w).$$

In particular, $R[\omega_RX]/\mathfrak{k}$ is a module finite extension of R , so it is also a complete local domain by [Co46, Theorem 7]. It is easily verified that $\mathfrak{m}_R + \omega_R a^{1/2}$ is the unique maximal ideal of $R[\omega_RX]/\mathfrak{k}$.

- (iv) Since $R \rightarrow R[\omega_RX]/\mathfrak{k}$ is a module finite extension in view of part (iii) (thus an integral extension by [Ma89, Theorem 9.1(i)]), Lemma 2.21 implies that R satisfies the CE Property provided that $R[\omega_RX]/\mathfrak{k}$ does so. Here we use that $\dim(R[\omega_RX]/\mathfrak{k}) = \dim(R) = n$ by [Ma89, Exercise 9.2, page 69] and that $\mathfrak{m}_R(R[\omega_RX]/\mathfrak{k})$ is primary to the maximal ideal of $R[\omega_RX]/\mathfrak{k}$ by [Ma89, Lemma 2, page 66]. Thus any system of parameters for R is also a system of parameters for $R[\omega_RX]/\mathfrak{k}$.
- (v) By [Ta17, Lemma 2.5(ii)], $R[\omega_RX]/\mathfrak{k}$ is a locally complete intersection in codimension 1.
- (vi) The complete local domain $R[\omega_RX]/\mathfrak{k}$ is quasi-Gorenstein. The proof of this fact is written in the proof of [Ta17, Proposition 2.7].

In view of parts (iv), (v) and (vi), we can replace R by the n -dimensional domain $R[\omega_RX]/\mathfrak{k}$ and we hereafter assume that R is a complete quasi-Gorenstein local domain that is a locally complete intersection in codimension ≤ 1 ⁵. Here, we also remark that R is not normal, because our involved quasi-Gorenstein domain extension loses the normal property even though it is a locally complete intersection in codimension ≤ 1 .

Step 2: The generic linkage and the setting. By the previous arguments, we can assume that R is a (non-Cohen-Macaulay) complete quasi-Gorenstein domain which is a locally complete intersection in codimension ≤ 1 . Then the Huneke and Ulrich generic linkage is applied to enable us to get a prime almost complete intersection ideal \mathfrak{a} of an excellent regular local ring A such that A/\mathfrak{a} is a locally complete intersection in codimension ≤ 1 and it is linked to a trivial deformation of R . Namely, following [HuU185] and [U184], we proceed as follows.

We assume that $R := B/\mathfrak{b}'$ where (B, \mathfrak{m}_B, K_B) is a complete regular local ring and that \mathfrak{b}' is a grade g ideal of B which is generated by u elements b'_1, \dots, b'_u . Then we consider the indeterminates

$$\mathbf{X} := X_{1,1}, \dots, X_{1,u}, X_{2,1}, \dots, X_{2,u}, \dots, X_{g,1}, \dots, X_{g,u}$$

over B and the sequence

$$a_1 := \sum_{j=1}^u X_{1,j} b'_j, \quad a_2 := \sum_{j=1}^u X_{2,j} b'_j, \quad \dots, \quad a_g := \sum_{j=1}^u X_{g,j} b'_j.$$

Then a_1, \dots, a_g is a maximal regular sequence contained in $\mathfrak{b}'B[\mathbf{X}]$ (see [Hoch73]). Set

$$\mathfrak{a}' := (a_1, \dots, a_g) : \mathfrak{b}'B[\mathbf{X}].$$

It is said that \mathfrak{a}' is a generic linkage of \mathfrak{b}' ([HuU185, Definition 2.3]). Since $B/\mathfrak{b}' \rightarrow B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}/\mathfrak{b}'B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}$ is a flat local homomorphism with regular closed fiber $K_B[\mathbf{X}]_{(\mathbf{X})}$, so $B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}/\mathfrak{b}'B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}$ is also (a non-Cohen-Macaulay) quasi-Gorenstein domain by [AoGo85, Theorem 4.1]⁶. Consequently, $\mathfrak{a}'B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}$ is an unmixed almost complete intersection ideal by Lemma 2.9(i). By virtue of [HuU185, Proposition

⁵In the case where R is a Cohen-Macaulay local domain with canonical ideal ω_R , such an R -algebra structure on $R \oplus \omega_R$ is considered in [En03, Definition 1.1 and Proposition 1.2] where this R -algebra is called as a pseudocanonical double cover of R , and it is proved that the pseudocanonical double cover is a Gorenstein local ring. However, the author of the present paper became aware of this R -algebra structure on $R \oplus \omega_R$ by his own experience and later he found a similar construction in [En03].

⁶In view of [TaTu18, Corollary 2.8], this quasi-Gorenstein property can also be deduced by an iterated use of Lemma 2.14. Also, one can deduce this fact from Lemma 2.9(ii), because the faithfully flat extension $B \rightarrow B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}$ preserves linkage of ideals, as well as the unmixedness and the almost complete intersection property of ideals.

2.6] \mathfrak{a}' is a prime ideal, and $B[\mathbf{X}]/\mathfrak{a}'$ is a locally complete intersection in codimension ≤ 1 in light of [HuUl85, Proposition 2.9(b)]. Hence $\mathfrak{a}'B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}$ is also a prime ideal whose residue ring is a locally complete intersection in codimension ≤ 1 . Set

$$(5.1) \quad (A, \mathfrak{m}_A) := B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}, \quad \mathfrak{c} := (a_1, \dots, a_g)A, \quad \mathfrak{b} := \mathfrak{b}'A, \quad \mathfrak{a} := \mathfrak{a}'A \stackrel{\text{Lemma 2.9(i)}}{=} \mathfrak{c} + (h), \quad (\text{for some } h).$$

So A is an excellent regular local ring, and $\mathfrak{a}, \mathfrak{b}$ are linked ideals of A over the complete intersection \mathfrak{c} (see Lemma 2.9(i)).

Step 3: Collecting some depth identities. Since A/\mathfrak{b} is a trivial deformation of $R = B/\mathfrak{b}'$ (by the image of \mathbf{X} in A/\mathfrak{b}), so setting $m := \dim(A/\mathfrak{b}) (= \dim(A/\mathfrak{a}))$

$$(5.2) \quad \text{depth}(A/\mathfrak{b}) = \dim(A/\mathfrak{b}) - \dim(R) + \text{depth}(R) \geq m - n + 2$$

($R (\cong \omega_R)$ is quasi-Gorenstein and it has depth ≥ 2 because the canonical module is always (S_2) and $\dim(R) \geq 3$ by our assumption). Consequently,

$$(5.3) \quad \text{depth}_A(\omega_{A/\mathfrak{a}}) \stackrel{\text{Eq. (2.2)}}{=} \text{depth}_A((\mathfrak{c} : \mathfrak{a})/\mathfrak{c}) = \text{depth}_A(\mathfrak{b}/\mathfrak{c}) = \text{depth}(A/\mathfrak{b}) + 1 \geq m - n + 3$$

where the third equality follows from $0 \rightarrow \mathfrak{b}/\mathfrak{c} \rightarrow A/\mathfrak{c} \rightarrow A/\mathfrak{b} \rightarrow 0$ and the fact that A/\mathfrak{c} is complete intersection (thus Cohen-Macaulay) while A/\mathfrak{b} is not Cohen-Macaulay. Furthermore

$$(5.4) \quad \text{depth}(A/\mathfrak{a}) = \text{depth}(A/\mathfrak{b}) - 1 \geq m - n + 1.$$

in view of display (5.2) and Lemma 3.1(i).

Step 4: Proving the CE Property for A/\mathfrak{a} . It suffices to find a regular sequence $\mathbf{y} := y_1, \dots, y_{m-n+1}$ of A/\mathfrak{a} such that

$$(5.5) \quad y_{i+1} \notin \bigcup_{\mathfrak{p} \in \text{Att}(H_{\mathfrak{m}_A}^{m-i-1}(A/(\mathfrak{a}, y_1, \dots, y_i)))} \mathfrak{p}$$

for each $0 \leq i \leq m - n$. Because then $\dim(A/(\mathfrak{a}, \mathbf{y})) = n - 1$ implying that $A/(\mathfrak{a}, \mathbf{y})$ satisfies the CE property by the induction hypothesis and then A/\mathfrak{a} satisfies the CE Property in view of Lemma 2.20(ii).

We prove the existence of the sequence \mathbf{y} by induction. By display (5.3) we have $\text{depth}(\omega_{A/\mathfrak{a}}) \geq 2$, so Lemma 2.12 implies that $\mathfrak{m}_A \notin \text{Att}(H_{\mathfrak{m}_A}^{m-1}(A/\mathfrak{a}))$ and we can pick some

$$y_1 \in \mathfrak{m}_A \setminus \left(\left(\bigcup_{\mathfrak{p} \in \text{Att}(H_{\mathfrak{m}_A}^{m-1}(A/(\mathfrak{a})))} \mathfrak{p} \right) \cup \mathfrak{b} \cup \left(\bigcup_{\mathfrak{p} \in \text{ass}(a_1, \dots, a_g)} \mathfrak{p} \right) \right).$$

Denoting the residue class of y_1 in A/\mathfrak{a} with the same notation y_1 , it is automatically a regular element as A/\mathfrak{a} is a domain. Moreover, from the exact sequence $0 \rightarrow A/\mathfrak{b} \xrightarrow{h} A/\mathfrak{c} \rightarrow A/\mathfrak{a} \rightarrow 0$ (see display (5.1) for the notation h) we get the exact sequence

$$0 \xrightarrow{y_1 \text{ is regular}} \text{Tor}_1^A(A/\mathfrak{a}, A/(y_1)) \rightarrow A/(\mathfrak{b}, y_1) \xrightarrow{h} A/(\mathfrak{c}, y_1) \rightarrow A/(\mathfrak{a}, y_1) \rightarrow 0$$

which implies that $(\mathfrak{c}, y_1) : (\mathfrak{a}, y_1) \stackrel{\mathfrak{a}=\mathfrak{c}+(h)}{=} (\mathfrak{c}, y_1) : h = (\mathfrak{b}, y_1)$. Therefore

$$\omega_{A/(\mathfrak{a}, y_1)} \stackrel{\text{Eq. (2.2)}}{=} ((\mathfrak{c}, y_1) : (\mathfrak{a}, y_1)) / (\mathfrak{c}, y_1) = (\mathfrak{b}, y_1) / (\mathfrak{c}, y_1)$$

(note that (\mathfrak{c}, y_1) is a (maximal) complete intersection ideal contained in (\mathfrak{a}, y_1) by our choice of y_1). In particular,

$$\begin{aligned}
\text{depth}(\omega_{A/(\mathfrak{a}, y_1)}) &= \text{depth}((\mathfrak{b}, y_1)/(\mathfrak{c}, y_1)) \\
&= \text{depth}(A/(\mathfrak{b}, y_1)) + 1 \quad ((\mathfrak{b}, y_1)/(\mathfrak{c}, y_1) \text{ is the first syzygy of } \underbrace{A/(\mathfrak{b}, y_1)}_{\text{non-CM}} \text{ over } \underbrace{A/(\mathfrak{c}, y_1)}_{\text{c.i.}}) \\
&= \text{depth}(A/\mathfrak{b}) \quad (y_1 \notin Z(A/\mathfrak{b})) \\
&\geq m - n + 2 \quad \text{Eq. (5.2)}.
\end{aligned}$$

Suppose, inductively, that for some $1 \leq j < m - n + 1$ we have found a sequence y_1, \dots, y_j in A which forms a regular sequence on each of A/\mathfrak{c} , A/\mathfrak{b} and A/\mathfrak{a} such that it satisfies display (5.5) for each $0 \leq i < j$, such that $\text{depth}(\omega_{A/(\mathfrak{a}, y_1, \dots, y_j)}) \geq m - n + 3 - j$, and finally such that

$$(\mathfrak{c}, y_1, \dots, y_j) : (\mathfrak{a}, y_1, \dots, y_j) = (\mathfrak{c}, y_1, \dots, y_j) : h = (\mathfrak{b}, y_1, \dots, y_j).$$

Since $j < m - n + 1$, so $m - n + 3 - j > 2$. Therefore, again Lemma 2.12 implies that $\mathfrak{m}_A \notin \text{Att}(H_{\mathfrak{m}_A}^{m-j-1}(A/(\mathfrak{a}, y_1, \dots, y_j)))$ and we may, and we do, pick some $y_{j+1} \in \mathfrak{m}_A$ such that y_{j+1} does not belong to

$$\left(\bigcup_{\mathfrak{p} \in \text{Att}(H_{\mathfrak{m}_A}^{m-j-1}(A/(\mathfrak{a}, y_1, \dots, y_j)))} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in \text{ass}(\mathfrak{b}, y_1, \dots, y_j)} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in \text{ass}(\mathfrak{c}, y_1, \dots, y_j)} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in \text{ass}(\mathfrak{a}, y_1, \dots, y_j)} \mathfrak{p} \right).$$

This is possible in view of displays (5.2) and (5.4). Therefore the exact sequence

$$0 \rightarrow A/((\mathfrak{c}, y_1, \dots, y_j) : h) \xrightarrow{h} A/(\mathfrak{c}, y_1, \dots, y_j) \rightarrow A/(\mathfrak{a}, y_1, \dots, y_j) \rightarrow 0$$

yields the exact sequence

$$0 \rightarrow A/((\mathfrak{c}, y_1, \dots, y_j) : h, y_{j+1}) \xrightarrow{h} A/(\mathfrak{c}, y_1, \dots, y_{j+1}) \rightarrow A/(\mathfrak{a}, y_1, \dots, y_{j+1}) \rightarrow 0$$

because $\text{Tor}_1^A(A/(\mathfrak{a}, y_1, \dots, y_j), A/(y_{j+1})) = 0$ by our choice of y_{j+1} . From this and our induction hypothesis, we get

$$\begin{aligned}
\text{depth}(\omega_{A/(\mathfrak{a}, y_1, \dots, y_{j+1})}) &= \text{depth}((\mathfrak{b}, y_1, \dots, y_{j+1})/(\mathfrak{c}, y_1, \dots, y_{j+1})) \quad (\text{display (2.2)}) \\
&= \text{depth}(A/(\mathfrak{b}, y_1, \dots, y_{j+1})) + 1 \\
&= \text{depth}(A/(\mathfrak{b}, y_1, \dots, y_j)) \quad (y_{j+1} \notin Z(A/(\mathfrak{b}, y_1, \dots, y_j))) \\
&= \text{depth}((\mathfrak{b}, y_1, \dots, y_j)/(\mathfrak{c}, y_1, \dots, y_j)) - 1 \\
&= \text{depth}(\omega_{A/(\mathfrak{a}, y_1, \dots, y_j)}) - 1 \\
&\geq m - n + 3 - (j + 1).
\end{aligned}$$

So our inductive argument is done, the desired regular sequence y_1, \dots, y_{m-n+1} exists and the CE Property holds for A/\mathfrak{a} .

Step 5: Proving the CE Property for $A[at, t^{-1}]_{\mathfrak{M}}$ in general, and thus the Strong Canonical Element Property for $A[at, t^{-1}]_{\mathfrak{M}}$ under the validity of the assertion (i). By Step 4 together with Lemma 2.14 and Lemma 2.20(ii), $(A/\mathfrak{a}[X])_{(\mathfrak{m}_A, X)}$ also satisfies the CE Property. From this fact together with Proposition 3.6(iii) we conclude that

$$(5.6) \quad A[at, t^{-1}]_{\mathfrak{M}}/(a_1 t, \dots, a_g t, ht) \cong ((A/\mathfrak{a})[X])_{(\mathfrak{m}_A, X)}$$

satisfies the CE Property. Since $\text{height}_A(\mathfrak{a}) \stackrel{\text{Remark 2.8}}{=} \text{height}_A(\mathfrak{b}) = \text{height}_B(\mathfrak{b}') = g$, so

$$\dim((A/\mathfrak{a}[X])_{(\mathfrak{m}_A, X)}) = \dim(A) - g + 1$$

and also

$$\dim(A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_gt)) = \dim(A) + 1 - g$$

as a_1t, \dots, a_gt is a (homogeneous) regular sequence on $A[\mathbf{a}t, t^{-1}]$ by Proposition 3.6(iv). So, display (5.6) shows that $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_gt)$ has the same dimension as its quotient $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_gt, ht)$, implying that the CE Property holds for $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_gt)$ by Lemma 2.21. But by Proposition 3.6(v), the regular sequence a_1t, \dots, a_gt satisfies the condition

$$a_{i+1}t \notin \bigcup_{\mathfrak{p} \in \text{Att}(H_{\mathfrak{M}}^{d-i}(A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}/(a_1t, \dots, a_it)))} \mathfrak{p}$$

for each $0 \leq i \leq g-1$. In particular, a repeated use of Lemma 2.20(ii) implies that $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}$ also fulfills the CE Property. Thus, since it is already known that $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}$ is an excellent factorial domain and it is a homomorphic image of a regular local ring, so from the validity of assertion (i) we can conclude that $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}$ satisfies the Strong Canonical Element Property.

Step 6: Collecting some sequences and identities: Now we turn our attention to $A[\mathbf{a}t, t^{-1}]_{ht}$. Let \mathbf{z}'' be any system of parameters for A/\mathfrak{a} , and lift it to a system of parameters \mathbf{z}' for A/\mathfrak{c} (such a lift of \mathbf{z}'' can be found applying the Davis' prime avoidance lemma as $\dim(A/\mathfrak{c}) = \dim(A/\mathfrak{a})$). We fix a lift of \mathbf{z}' in A , that is denoted by abuse of notation, again with \mathbf{z}' . Set $\mathfrak{P} := (\mathfrak{m}_A, a_1t, \dots, a_gt, t^{-1})$, that is a height $d := \dim(A)$ prime ideal (Proposition 3.6(i)). We claim that, $\mathbf{z}', a_1t, \dots, a_gt$ is a sequence of length d such that

$$(5.7) \quad \sqrt{(\mathbf{z}', a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]_{ht}} = \mathfrak{P}A[\mathbf{a}t, t^{-1}]_{ht}.$$

Namely, we have $(\mathbf{z}', a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}] = (\mathbf{z}', \mathfrak{c}, a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]$ consequently it contains $(\mathfrak{m}_A^u, a_1t, \dots, a_gt)$ for some u (\mathbf{z}' forms a system of parameters for A/\mathfrak{c}). So

$$(5.8) \quad \sqrt{(\mathbf{z}', a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]} = \sqrt{(\mathfrak{m}_A, a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]}.$$

Moreover, $(\mathfrak{m}_A, a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]_{ht} = \mathfrak{P}A[\mathbf{a}t, t^{-1}]_{ht}$ because $t^{-1} = h/ht \in (\mathfrak{m}_A, a_1t, \dots, a_gt)A[\mathbf{a}t, t^{-1}]_{ht}$. Hence display (5.7) holds. Set

$$\mathbf{z} := \mathbf{z}', a_1t/ht, \dots, a_gt/ht,$$

a sequence of elements of degree zero of $A[\mathbf{a}t, t^{-1}]_{ht}$.

In addition to \mathbf{z} , there is a possibly inhomogeneous sequence $\mathbf{f} := f_1, \dots, f_d$ of elements of \mathfrak{P} such that

$$\sqrt{(f_1, \dots, f_d, ht)A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}} = (\mathfrak{M}),$$

so f_1, \dots, f_d, ht forms a system of parameters for $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}$ (this is possible as $\mathfrak{M} = (\mathfrak{P}, ht)$ and ht is a parameter element of the domain $A[\mathbf{a}t, t^{-1}]_{\mathfrak{M}}$).

Step 7: Proving an analogue of the CE Property for $A[\mathbf{a}t, t^{-1}]_{ht}$ and the sequence f_1, \dots, f_d . Let $(F_{\bullet}, \partial_{\bullet}^F)$ be a minimal graded free resolution of $A[\mathbf{a}t, t^{-1}]/\mathfrak{P}$. Fix some chain map

$$\phi_{\bullet} : K_{\bullet}(\mathbf{f}; A[\mathbf{a}t, t^{-1}]) \rightarrow F_{\bullet}$$

lifting the epimorphism $A[\mathbf{a}t, t^{-1}]/(\mathbf{f}) \rightarrow A[\mathbf{a}t, t^{-1}]/\mathfrak{P}$. Passing from $A[\mathbf{a}t, t^{-1}]$ to its localization $A[\mathbf{a}t, t^{-1}]_{ht}$, this induces the chain map

$$(\phi_{\bullet})_{ht} : K_{\bullet}(\mathbf{f}; A[\mathbf{a}t, t^{-1}]_{ht}) \rightarrow F_{\bullet ht}$$

for which we claim that (under the validity of either assertion (i) or assertion (ii) of the statement)

$$(5.9) \quad \forall v \in \mathbb{N}, \quad f_1^{v-1} \dots f_d^{v-1} \phi_d(1)/1 + (\text{im}(\partial_{d+1}^F))_{ht} \notin (f_1^v, \dots, f_d^v) \text{syz}_d^{F_{\bullet ht}}(A[\mathbf{a}t, t^{-1}]_{ht}/(\mathfrak{P})).$$

To get a contradiction, suppose the opposite of display (5.9) for some $v \in \mathbb{N}$. Then,

$$(ht)^{m'} f_1^{v-1} \dots f_d^{v-1} \phi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) \in (f_1^v, \dots, f_d^v) \text{syz}_d^{F_\bullet}(A[\mathfrak{at}, t^{-1}]/\mathfrak{P})$$

for some $m' \in \mathbb{N}_0$.

If assertion (ii) holds, then $A[\mathfrak{at}, t^{-1}]_{\mathfrak{P}}$ satisfies the Canonical Element Conjecture and we get a contradiction with display (5.9) after localizing at \mathfrak{P} (because f_1, \dots, f_d forms a system of parameters for $A[\mathfrak{at}, t^{-1}]_{\mathfrak{P}}$ and ht becomes invertible in $A[\mathfrak{at}, t^{-1}]_{\mathfrak{P}}$). So let us assume that assertion (i) holds.

Setting $w := m' + v$

$$(5.10) \quad (ht)^{w-1} f_1^{w-1} \dots f_d^{w-1} \phi_d(1) + \text{im}(\partial_{d+1}^{F_\bullet}) \in (f_1^w, \dots, f_d^w) \text{syz}_d^{F_\bullet}(A[\mathfrak{at}, t^{-1}]/\mathfrak{P}).$$

Let G_\bullet be the mapping cone of

$$(ht)^w \cdot \text{id}_{F_\bullet} : F_\bullet \rightarrow F_\bullet,$$

so G_\bullet is a free resolution of $A[\mathfrak{at}, t^{-1}]/(\mathfrak{P}, (ht)^w)$ in view of Remark 4.2. Also the mapping cone of

$$(ht)^w \cdot \text{id}_{K_\bullet(\mathbf{f}; A[\mathfrak{at}, t^{-1}])} : K_\bullet(\mathbf{f}; A[\mathfrak{at}, t^{-1}]) \rightarrow K_\bullet(\mathbf{f}; A[\mathfrak{at}, t^{-1}])$$

coincides with the Koszul complex $K_\bullet(\mathbf{f}, (ht)^w; A[\mathfrak{at}, t^{-1}])$ (see [ScSi18, Definition and Observations 5.2.1], and also [ScSi18, 1.5.1] for the used notation). Then ϕ_\bullet extends to a chain map on the mapping cones, resulting in the chain map

$$\psi_\bullet : K_\bullet(\mathbf{f}, (ht)^w; A[\mathfrak{at}, t^{-1}]) \rightarrow G_\bullet, \quad \psi_i := (\phi_i, \phi_{i-1})$$

which lifts the natural epimorphism $A[\mathfrak{at}, t^{-1}]/(\mathbf{f}, (ht)^w) \twoheadrightarrow A[\mathfrak{at}, t^{-1}]/(\mathfrak{P}, (ht)^w)$. Then $\psi_{d+1} = \phi_d$, and the relation in display (5.10) yields

$$(5.11) \quad (ht)^{w-1} f_1^{w-1} \dots f_d^{w-1} \psi_{d+1}(1) + \text{im}(\partial_{d+2}^{G_\bullet}) \in ((ht)^w, f_1^w, \dots, f_d^w) \text{syz}_{d+1}^{G_\bullet}(A[\mathfrak{at}, t^{-1}]/(\mathfrak{P}, (ht)^w)).$$

Namely, by display (5.10) we have

$$(ht)^{w-1} f_1^{w-1} \dots f_d^{w-1} \phi_d(1) = \sum_{i=1}^d f_i^w \alpha_i + \partial_{d+1}^{F_\bullet}(\beta), \quad \alpha_1, \dots, \alpha_d \in F_d, \beta \in F_{d+1}.$$

So, it is easily seen that

$$(ht)^{w-1} f_1^{w-1} \dots f_d^{w-1} \psi_{d+1}(1) = (ht)^w(-\beta, 0) + \sum_{i=1}^d f_i^w(0, \alpha_i) + \partial_{d+2}^{G_\bullet}(0, -\beta).$$

Applying Lemma 2.2 and composing ψ_\bullet with the natural chain map

$$\eta_w : K_\bullet((ht)^w, f_1^w, \dots, f_d^w; A[\mathfrak{at}, t^{-1}]) \rightarrow K_\bullet((ht)^w, \mathbf{f}; A[\mathfrak{at}, t^{-1}]),$$

in view of display (5.11), we get the chain map $\Delta_\bullet : K_\bullet((ht)^w, f_1^w, \dots, f_d^w; A[\mathfrak{at}, t^{-1}]) \rightarrow G_\bullet$ lifting $A[\mathfrak{at}, t^{-1}]/(\mathbf{f}^w, (ht)^w) \twoheadrightarrow A[\mathfrak{at}, t^{-1}]/(\mathfrak{P}, (ht)^w)$ such that

$$(ht)^{w-1} \Delta_{d+1}(1) + \text{im}(\partial_{d+2}^{G_\bullet}) \in ((ht)^w, f_1^w, \dots, f_d^w) \text{syz}_{d+1}^{G_\bullet}(A[\mathfrak{at}, t^{-1}]/(\mathfrak{P}, (ht)^w)).$$

However, by Step 5 and assertion (i) in the statement of the theorem, we know that $A[\mathfrak{at}, t^{-1}]_{\mathfrak{M}}$ satisfies the Strong Canonical Element Property and so this is a contradiction, completing this step.

Step 8: Proving an analogue of the CE Property for $A[\mathfrak{at}, t^{-1}]_{ht}$ and the sequence \mathbf{z} . Now, we claim that the CE Property mentioned in display (5.9) for the possibly inhomogeneous sequence f_1, \dots, f_d in $A[\mathfrak{at}, t^{-1}]_{ht}$ implies the similar CE Property for the homogeneous sequence \mathbf{z} in $A[\mathfrak{at}, t^{-1}]_{ht}$. Namely, we consider an $A[\mathfrak{at}, t^{-1}]_{ht}$ -graded free resolution $(P_\bullet, \partial_\bullet^P)$ of

$$A[\mathfrak{at}, t^{-1}]_{ht}/(\mathfrak{m}_A, a_1 t/ht, \dots, a_g t/ht)$$

such that there is no shift involved in each graded free module in the graded free resolution P_\bullet . This no-shift-property is accessible because $A[\mathfrak{a}t, t^{-1}]_{ht}$ has an invertible element of degree 1 and all syzygies are generated in degree 0.

Let $\Psi_\bullet : K_\bullet(\mathbf{z}; A[\mathfrak{a}t, t^{-1}]_{ht}) \rightarrow P_\bullet$ be an arbitrary homogeneous chain map lifting the natural epimorphism $A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathbf{z}) \rightarrow A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)$. We show that

$$(5.12) \quad \forall v \in \mathbb{N}, z_1^{v-1} \dots z_d^{v-1} \Psi_d(1) + \text{im}(\partial_{d+1}^{P_\bullet}) \notin (z_1^v, \dots, z_d^v)_{\text{syzy}_d^{P_\bullet}}(A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)).$$

If not, there is some v such that display (5.12) is violated. Let Ψ'_\bullet be the composition of the chain maps

$$K_\bullet(\mathbf{z}^v; A[\mathfrak{a}t, t^{-1}]_{ht}) \xrightarrow{\eta_v} K_\bullet(\mathbf{z}; A[\mathfrak{a}t, t^{-1}]_{ht}) \xrightarrow{\Psi_\bullet} P_\bullet$$

where η_v is as in Lemma 2.2(ii), so $\Psi'_d(1) = z_1^{v-1} \dots z_d^{v-1} \Psi_d(1)$ by Lemma 2.2(ii). Then, by our hypothesis we get

$$\Psi'_d(1) + \text{im}(\partial_{d+1}^{P_\bullet}) \in (z_1^v, \dots, z_d^v)_{\text{syzy}_d^{P_\bullet}}(A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)).$$

Hence,

$$\Psi'_d(1) \equiv \sum_{i=1}^d z_i^v \alpha_i \pmod{\text{im} \partial_{d+1}^{P_\bullet}}$$

for some $\alpha_1, \dots, \alpha_d \in P_d$.

Set $\Psi''_i := \Psi'_i$ for each $i < d-1$, $\Psi''_d := 0$ and assume that Ψ''_{d-1} is given by

$$\Psi''_{d-1}(e_1 \wedge \dots \wedge \widehat{e}_i \wedge \dots \wedge e_d) = \Psi'_{d-1}(e_1 \wedge \dots \wedge \widehat{e}_i \wedge \dots \wedge e_d) - \partial_d^{P_\bullet}((-1)^{i+1} \alpha_i).$$

By definition, it is clear that

$$\partial_{d-1}^{P_\bullet} \circ \Psi''_{d-1} = \partial_{d-1}^{P_\bullet} \circ \Psi'_{d-1} = \Psi'_{d-2} \circ \partial_{d-1}^{K_\bullet(\mathbf{z}^v; A[\mathfrak{a}t, t^{-1}]_{ht})} = \Psi''_{d-2} \circ \partial_{d-1}^{K_\bullet(\mathbf{z}^v; A[\mathfrak{a}t, t^{-1}]_{ht})}.$$

It is also easily verified that $\Psi''_{d-1} \circ \partial_d^{K_\bullet(\mathbf{z}^v; A[\mathfrak{a}t, t^{-1}]_{ht})} = 0$. Consequently, we get another chain map

$$\Psi''_\bullet : K_\bullet(\mathbf{z}^v; A[\mathfrak{a}t, t^{-1}]_{ht}) \rightarrow P_\bullet$$

lifting $A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathbf{z}^v) \xrightarrow{\text{nat. epi.}} A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)$ such that $\Psi''_d = 0$. From display (5.7) and the fact that $t^{-1} = h/ht \in (\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)$ we deduce that

$$(f_1, \dots, f_d)A[\mathfrak{a}t, t^{-1}]_{ht} \subseteq (\mathfrak{P}) = (\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht) = \sqrt{(\mathbf{z})},$$

implying that $(f_1^w, \dots, f_d^w) \subseteq (z_1^v, \dots, z_d^v)$ for some w . Thus by Lemma 2.2(i) there exists a chain map of Koszul complexes

$$\lambda_\bullet : K_\bullet(f_1^w, \dots, f_d^w; A[\mathfrak{a}t, t^{-1}]_{ht}) \rightarrow K_\bullet(z_1^v, \dots, z_d^v; A[\mathfrak{a}t, t^{-1}]_{ht})$$

lifting the natural epimorphism on the 0-th homologies. Then the composition

$$\Theta_\bullet := \Psi''_\bullet \circ \lambda_\bullet : K_\bullet(f_1^w, \dots, f_d^w; A[\mathfrak{a}t, t^{-1}]_{ht}) \rightarrow P_\bullet$$

lifts the natural epimorphism on the 0-homologies while $\Theta_d(1) = 0$. Then since P_\bullet and $(F_\bullet)_{ht}$ are both free resolutions of $A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{P})$, so there is a chain map $\pi_\bullet : P_\bullet \rightarrow (F_\bullet)_{ht}$ lifting the identity map on $A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{P})$, by which we get a chain map

$$\delta_\bullet := \pi_\bullet \circ \Theta_\bullet : K_\bullet(f_1^w, \dots, f_d^w; A[\mathfrak{a}t, t^{-1}]_{ht}) \rightarrow (F_\bullet)_{ht}$$

which lifts the natural epimorphism on the 0-homologies and $\delta_d(1) = 0$. However, the composited chain map (the chain map $(\phi_\bullet)_{ht}$ is as in Step 7)

$$w\Phi_\bullet : K_\bullet(f_1^w, \dots, f_d^w; A[\mathfrak{a}t, t^{-1}]_{ht}) \xrightarrow{w\zeta} K_\bullet(\mathbf{f}; A[\mathfrak{a}t, t^{-1}]_{ht}) \xrightarrow{(\phi_\bullet)_{ht}} (F_\bullet)_{ht}$$

also lifts

$$A[\mathfrak{a}t, t^{-1}]_{ht}/(f_1^w, \dots, f_d^w) \xrightarrow{\text{nat. epi.}} A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{P})$$

where ${}_w\zeta$ is as in Lemma 2.2(ii). Then, we get

$$\begin{aligned} f_1^{w-1} \dots f_d^{w-1} \phi_d(1)/1 + (\text{im}(\partial_{d+1}^{F_\bullet}))_{ht} &= {}_w\Phi_d(1) + (\text{im}(\partial_{d+1}^{F_\bullet}))_{ht} && \text{(Lemma 2.2(ii))} \\ &= {}_w\Phi_d(1) - \underbrace{\delta_d(1)}_{=0} + (\text{im}(\partial_{d+1}^{F_\bullet}))_{ht} \\ &\in (f_1^w, \dots, f_d^w) \text{syz}_d^{F_\bullet, ht}(A[\mathfrak{a}t, t^{-1}]_{ht}/(\mathfrak{P})) && \text{(Lemma 2.1(ii)).} \end{aligned}$$

This contradicts display (5.9).

Step 9: Proving an analogue of the CE Property for $(A[\mathfrak{a}t, t^{-1}]_{ht})_{[0]}$ and the sequence \mathbf{z} . We set $Q := (A[\mathfrak{a}t, t^{-1}]_{ht})_{[0]}$. Since there is no shift involved in each graded free module in the graded free resolution P_\bullet , so the degree zero part of each graded free module in the free resolution P_\bullet is a finite direct sum of Q , hence a free Q -module. Consequently, $(P_\bullet)_{[0]}$ is a free resolution of $Q/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)$.

Furthermore, $(K_\bullet(\mathbf{z}; A[\mathfrak{a}t, t^{-1}]_{ht}))_{[0]} = K_\bullet(\mathbf{z}; Q)$. Thus, Ψ_\bullet induces $(\Psi_\bullet)_{[0]} : K_\bullet(\mathbf{z}; Q) \rightarrow (P_\bullet)_{[0]}$.

Moreover, the analogue to display (5.12) in Q holds. Namely, $(\Psi_\bullet)_{[0]} : K_\bullet(\mathbf{z}; Q) \rightarrow (P_\bullet)_{[0]}$ is a chain map lifting the natural surjection on 0-th homologies such that

$$(5.13) \quad \forall v \in \mathbb{N}, \quad z_1^{v-1} \dots z_d^{v-1} \underbrace{(\Psi_d)_{[0]}(1)}_{=\Psi_d(1)} + \text{im}(\partial_{d+1}^{(P_\bullet)_{[0]}}) \notin (z_1^v, \dots, z_d^v) \text{syz}_d^{(P_\bullet)_{[0]}}(Q/(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)).$$

Step 10: Proving the CE Property for R . From Proposition 3.6(iv), it is easily deduced that $a_1t/ht, \dots, a_gt/ht$ is a regular sequence of Q . Moreover, from [HeSiVa83, Theorem 3.5]⁷ in conjunction with Lemma 3.1(ii) we deduce that \mathfrak{a} is an ideal of linear type, in other words, the Rees algebra $A[\mathfrak{a}t]$ of \mathfrak{a} coincides with the symmetric algebra $\text{Sym}_A(\mathfrak{a})$ of \mathfrak{a} . Thus recalling our notation in display (5.1), from

$$\begin{aligned} Q/(a_1t/ht, \dots, a_gt/ht) &= (A[\mathfrak{a}t, t^{-1}]_{ht})_{[0]}/(a_1t/ht, \dots, a_gt/ht) \\ &\cong (A[\mathfrak{a}t]_{ht})_{[0]}/(a_1t/ht, \dots, a_gt/ht) && A[\mathfrak{a}t]_{ht} \rightarrow A[\mathfrak{a}t, t^{-1}]_{ht} \text{ is an isomorphism} \\ &\cong (\text{Sym}_A(\mathfrak{a})_{ht})_{[0]}/(a_1t/ht, \dots, a_gt/ht) && \mathfrak{a} \text{ is an ideal of linear type} \\ &\cong A/(\mathfrak{c} : h) && \text{(Lemma 3.2(ii))} \\ &= A/\mathfrak{b}, \end{aligned}$$

we see that Q is a (possibly non-local) deformation of A/\mathfrak{b} . Also, $A/\mathfrak{b} = B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})}/\mathfrak{b}'B[\mathbf{X}]_{(\mathfrak{m}_B, \mathbf{X})} \cong R[\mathbf{X}]_{(\mathfrak{m}_R, \mathbf{X})}$ is a trivial deformation of R by the image of the indeterminates \mathbf{X} . Consequently, Q is a (possibly non-local) deformation of R .

In view of the above display, $(\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)Q$ is a maximal ideal of Q . Moreover, we have $\sqrt{\mathfrak{m}_A}Q = (\mathfrak{m}_A, a_1t/ht, \dots, a_gt/ht)Q$ by display (5.8). In view of these facts as well as display (5.13), we are in the situation of Lemma 2.20(i). Hence R satisfies the CE Property by Lemma 2.20(i) (Alternatively, one may first conclude from Lemma 2.20(i) as well as the above display that A/\mathfrak{b} satisfies the CE Property. Then, our desired conclusion follows from a second use of Lemma 2.20(i) this time with respect to the deformation A/\mathfrak{b} of $R = (A/\mathfrak{b})/(\mathbf{X})$, by replacing Q (respectively, $Q/\mathfrak{x}Q$) in the statement of the Lemma 2.20(i) with A/\mathfrak{b} (respectively, R)). \square

For the notion of Hochster's modification module, mentioned in the statement of the next corollary, we refer to [BrHe98, 8.3 Modifications and non degeneracy]. It turns out that Hochster's modification module is a big Cohen-Macaulay module precisely when it remains non-zero after tensoring with the residue

⁷Theorem 3.6 of the researchgate edition of [HeSiVa83].

field; this non-zero property follows from the existence of any balanced big Cohen-Macaulay module. Via the next corollary we show that concluding the big Cohen-Macaulayness of Hochster's modification module from the existence of a maximal Cohen-Macaulay complex, if possible, establishes the Balanced Big Cohen-Macaulay Module Conjecture by a characteristic free proof in general (see [IyMaScWa21, 4. MCM complexes] for the notion of maximal Cohen-Macaulay complexes). Alternatively, one may ask whether the existence of a maximal Cohen-Macaulay complex implies the existence of a closure operation satisfying the axioms of [Di10, Axioms 1.1]. See Remark 5.3(ii), for more comments on the assumptions in the statement of the next corollary.

Corollary 5.2. *If the big Cohen-Macaulayness of Hochster's modification module can be deduced from the existence of a maximal Cohen-Macaulay complex by a characteristic free proof, then the (Balanced) Big Cohen-Macaulay Module Conjecture can be settled by a characteristic free proof.*

Proof. We show that our hypothesis on deducing the big Cohen-Macaulayness of Hochster's modification module from the existence of a maximal Cohen-Macaulay complex implies that the Canonical Element Conjecture holds by a characteristic free proof, while the latter is equivalent to the existence of a maximal Cohen-Macaulay complex for any complete local ring by [IyMaScWa21, Proposition 4.3] and [Ho83, Theorem (4.3)]. Then a second use of our hypothesis in conjunction with Remark 4.18(i) and [IyMaScWa21, Proposition 4.3] settles the statement by a characteristic free proof.

To conclude the validity of the Canonical Element Conjecture by a characteristic free proof, in view of Theorem 5.1(ii), it suffices to show that any localization of a normal excellent local domain R satisfies the Canonical Element Conjecture provided R does. So let R be an excellent normal local domain which satisfies the Canonical Element Conjecture. Then \widehat{R} satisfies the Canonical Element Conjecture ([Ho83, Proposition (3.18)(b)]). Thus \widehat{R} admits a maximal Cohen-Macaulay complex by [IyMaScWa21, Proposition 4.3] and [Ho83, Theorem (4.3)], and so \widehat{R} , and thence R , admits a balanced big Cohen-Macaulay module by our hypothesis as well as Remark 4.18 (by a characteristic free proof). It then turns out that any localization of R admits a balanced big Cohen-Macaulay module by Theorem 4.17, a fortiori any localization of R satisfies the Canonical Element Conjecture as was to be proved. \square

Remark 5.3. It is perhaps appropriate to elucidate why we presented Theorem 4.17 with a longer proof than the proof of [BMPSTWW21, Proposition 2.11] and why we did not simply refer to the already existing result [BMPSTWW21, Proposition 2.11] in the literature? We reason this as follows:

- (i) Our Theorem 4.17 is a new result showing that the algebra structure of balanced big Cohen-Macaulay algebras over complete normal (or excellent normal) local domains is not required for their localizing property.
- (ii) The main result of Section 4, i.e. Corollary 4.19, can be deduced from [BMPSTWW21, Proposition 2.11] in place of Theorem 4.17. But, in contrast, Corollary 5.2 essentially requires Theorem 4.17 and does not follow from [BMPSTWW21, Proposition 2.11]. This is because in the statement of Corollary 5.2, we mention that Hochster's modification module is big Cohen-Macaulay (not Hochster's modification algebra), so we would need in its proof the localizing property of balanced big Cohen-Macaulay modules rather than balanced big Cohen-Macaulay algebras. For more comments, recall that the existence of a maximal Cohen-Macaulay complex (without an algebra structure) is equivalent to the Canonical Element Conjecture. Since such maximal Cohen-Macaulay complexes are not known to carry a DG-algebra structure (or any algebra structure), it seems unlikely (in the eyes of the author) to deduce the big Cohen-Macaulayness of Hochster's modification algebra from the existence of such a maximal Cohen-Macaulay complex. This is the reason that in the statement of Corollary 5.2 we assumed the deduction of big Cohen-Macaulayness of Hochster's modification

module (not algebra) from the existence of a maximal Cohen-Macaulay complex. Here, it seems also necessary to bring to the reader's attention that algebras of finite type over a field of characteristic zero admit a maximal Cohen-Macaulay complex carrying a DG-algebra structure (see [LyMaScWa21, Subsection 4.3]).

- (iii) As a main result of the present paper we showed that for obtaining a characteristic free proof of the Canonical Element Theorem it suffices to show that it is stable under the localization over excellent factorial (or excellent normal) domains. So perhaps providing other results in this direction as we did in Theorem 4.17 is promising that one can also establish the stability of the CE Property under the localization. As an immediate question in this direction, one may ask whether the proof of Theorem 4.17 can be modified appropriately to show that the maximal Cohen-Macaulay complexes behave well under the localization. An affirmative answer to this question establishes the Canonical Element Theorem with a characteristic free proof.

Remark 5.4. In [Ta22], we showed that the Monomial Theorem is equivalent to the assertion that for any (possibly non-unmixed) almost complete intersection ring (R, \mathfrak{m}) and any system of parameters \mathbf{x} of R ,

$$((\mathbf{x}) :_R \mathfrak{m})H_1(\mathbf{x}; R) = 0$$

(see [Ta22, Proposition 2.11]). Then a question (not a conjecture) is proposed in [Ta22]. Are all positive Koszul homologies $H_{i \geq 1}(\mathbf{x}; R)$ killed by $(\mathbf{x}) : \mathfrak{m}$ for any system of parameters \mathbf{x} of any almost complete intersection (R, \mathfrak{m}) (see [Ta22, Question 1.1] and [Ta22, Question 1.2])? We conclude with two comments regarding this question:

- We are aware of an example showing that the almost complete intersection hypothesis in [Ta22, Proposition 2.11] (and so in [Ta22, Question 1.1]) is necessary and can not be relaxed. More precisely, such an example violates the inclusion $((\mathbf{x}) :_R \mathfrak{m}) \subseteq 0 :_R H_1(\mathbf{x}; R)$ while this example still fulfills $(\mathbf{x}) \subsetneq 0 :_R H_1(\mathbf{x}; R)$ (here \mathbf{x} is a system of parameters for R where R is indeed not an almost complete intersection).
- As a positive result concerning [Ta22, Question 1.1 and Question 1.2], in [Ta]⁸ we showed that any almost complete intersection (R, \mathfrak{m}) admits a system of parameters \mathbf{x} as well as some $z \in ((\mathbf{x}) :_R \mathfrak{m}) \setminus (\mathbf{x})$ such that $zH_{i \geq 1}(\mathbf{x}; R) = 0$. In other words, any almost complete intersection (R, \mathfrak{m}) admits a system of parameters \mathbf{x} and an $\mathbf{x}R$ -socle element z (as above) for which the residual approximation complex $\mathcal{Z}_\bullet^+(\mathbf{x}, z; R)$ (which is a non-free finite complex) is acyclic with $H_0(\mathcal{Z}_\bullet^+(\mathbf{x}, z; R)) = R/\mathfrak{m}$.

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⁸The project [Ta] is in its very early stage and its ultimate version, if/when it is available in the future, can have a different title or/and can have more authors.

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