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Postulation of generic lines and one double line in $\mathbb{P}^n$ in view of generic lines and one multiple linear space

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Abstract

A well-known theorem by Hartshorne and Hirschowitz (in: Aroca, Buchweitz, Giusti, Merle (eds) Algebraic geometry. Lecture notes in mathematics, Springer, Berlin, 1982) states that a generic union $\mathbb{X} \subset \mathbb{P}^n$, $n \geq 3$, of lines has good postulation with respect to the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. So a question that naturally arises in studying the postulation of non-reduced positive dimensional schemes supported on linear spaces is the question whether adding a m-multiple c-dimensional linear space $m\mathbb{P}^c$ to $\mathbb{X}$ can still preserve it's good postulation, which means in classical language that, whether $m\mathbb{P}^c$ imposes independent conditions on the linear system $|\mathcal{I}_{\mathbb{X}}(d)|$. Recently, the case of $c = 0$, i.e., the case of lines and one m-multiple point, has been completely solved by several authors (Carlini et al. in Ann Sc Norm Super Pisa Cl Sci (5) XV:69–84, 2016; Aladpoosh and Ballico in Rend Semin Mat Univ Politec Torino 72(3–4):127–145, 2014; Ballico in Mediterr J Math 13(4):1449–1463, 2016) starting with Carlini–Catalisano–Geramita, while the case of $c > 0$ was remained unsolved, and this is what we wish to investigate in this paper. Precisely, we study the postulation of a generic union of s lines and one m-multiple linear space $m\mathbb{P}^c$ in $\mathbb{P}^n$, $n \geq c + 2$. Our main purpose is to provide a complete answer to the question in the case of lines and one double line, which says that the double line imposes independent conditions on $|\mathcal{I}_{\mathbb{X}}(d)|$ except for the only case $\{n = 4, s = 2, d = 2\}$. Moreover, we discuss an approach to the general case of lines and one m-multiple c-dimensional linear space, $(m \geq 2, c \geq 1)$, particularly, we find several exceptional such schemes, and we conjecture that these are the only exceptional ones in this family. Finally, we give some partial results in support of our conjecture.

Keywords Good postulation · Specialization · Degeneration · Double line · Double point · Generic union of lines · Sundial · Residual scheme · Hartshorne–Hirschowitz theorem · Castelnuovo's inequality

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1 Introduction

To understand the geometry of a closed subscheme X as an embedded scheme in $\mathbb{P}^n$, one of the first points of interest is considering the postulation problem, i.e. determining the number of conditions imposed by asking hypersurfaces of any degree to contain X. In terms of sheaf cohomology, we would like to know the rank of the restriction maps

$$\rho(d) : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(X, \mathcal{O}_X(d)).$$

We say that X has *maximal rank* or *good postulation* or *expected postulation* if $\rho(d)$ has maximal rank, i.e. it is either injective or surjective, for each $d \geq 0$. This amounts to saying that one or other of the integers $h^0(\mathcal{I}_X(d)),\ h^1(\mathcal{I}_X(d))$ is zero, and this shows that the property that X imposes independent conditions to degree d hypersurfaces can be interpreted cohomologically.

On the other hand, this classical problem is equivalent to computing the Hilbert function of X. Let $HF(X, d)$ be the Hilbert function of X in degree d, namely, $HF(X, d) = h^0(\mathcal{O}_{\mathbb{P}^n}(d)) - h^0(\mathcal{I}_X(d))$, i.e. the rank of $\rho(d)$. In order to determine the Hilbert function of X in some degree d, there is an expected value for it given by a naive count of conditions. This value is determined by assuming that X imposes independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of degree d hypersurfaces in $\mathbb{P}^n$, i.e.,

$$h^0(\mathcal{I}_X(d)) = \max\left\{h^0(\mathcal{O}_{\mathbb{P}^n}(d)) - h^0(\mathcal{O}_X(d)), 0\right\},$$

or equivalently

$$HF(X, d) = \min\{HP(X, d),\ HP(\mathbb{P}^n, d)\},$$

where $HP(X, d)$ is the Hilbert polynomial of X. In [7], the authors called a scheme X with such Hilbert function for all $d \geq 0$, has *bipolynomial Hilbert function*. It always holds $HF(X, d) \leq \min\{HP(X, d),\ HP(\mathbb{P}^n, d)\}$, so a natural question to ask is: when does this inequality turn into equality?

An important observation is that the postulation problem depends not only on the numerical data involved in it, but also on the position of the components of X. If X is in sufficiently general position one expects that X has bipolynomial Hilbert function, and therefore has good postulation, but this naive guess is in general false.

When we restrict our attention to the special class of schemes $X \subset \mathbb{P}^n$ which supported on unions of generic linear spaces, there is much interest in the postulation problem. In this situation it is noteworthy that the notions of good postulation and bipolynomial Hilbert function coincide. Original investigation mostly concentrated on the reduced cases (see e.g. [3,4,7,9,13,14,19,22]): if $\dim X = 0$, i.e. X is a generic collection of points in $\mathbb{P}^n$, it is well known that X has good postulation (see [18]); if $\dim X = 1$, we have a brilliant result due to Hartshorne and Hirschowitz, going back to 1981 [22], which states that a generic collection of lines in $\mathbb{P}^n$, $n \geq 3$, has good postulation; as soon as we go up to $\dim X > 1$, the postulation problem becomes more and more complicated. The extent of our ignorance in this situation is illustrated by the fact that the complete answer to the postulation problem even in two-dimensional case is not yet known (see [4,7] for a generic union of lines and a few planes, and [3] for a generic union of lines and a linear space).

On the other hand there is also a lot of interest on the postulation of non-reduced schemes supported on linear spaces: concerning the zero-dimensional case, i.e. fat points schemes, the postulation problem is a field of active research in algebraic geometry which has occupied researchers's minds for over a century, but, despite all the progress made on this problem, it is still very live and widely open in its generality (see e.g. [2,12,20], and also [11] for a survey of results, related conjectures and open questions); concerning the positive dimensional case, the postulation problem turns out to be far too complicated and giving a complete answer appears to be ambitious and quite difficult, this is why that it has never been systematically studied and in fact it had remained unsolved for a long time. Apparently the work of Carlini–Catalisano–Geramita [10] is a turning point in this story, which together with the recent papers [1] and [5] shows that a generic union X of s lines and one m-multiple point in $\mathbb{P}^n$, $n \geq 3$, always has good postulation in degree d except for the cases $\{n = 3, m = d, 2 \leq s \leq d\}$, (see for the proof: [10] in the case of $n \geq 4$, [1] in the case of $\{n = 3, m = 3\}$, and [5] in the case of $\{n = 3, m \geq 4\}$). As far as we know, this is the only complete knowledge of the postulation in the case of non-reduced positive dimensional schemes supported on linear spaces (for other related results see [15,16]). This paper was motivated by an attempt to go further in this direction, namely one may ask about the generic union of lines and one m-multiple line and one may hope that behaves well with respect to the postulation problem modulo a certain list of exceptions. The main result of this paper is solving the case of $m = 2$, i.e. the case of a generic union of lines and one double line. More precisely, we prove the following theorem:

Theorem 1.1 *Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one double line. Then X has good postulation, i.e.,*

$$h^0(\mathcal{I}_X(d)) = \max\left\{\binom{d+n}{n} - (nd+1) - s(d+1), 0\right\},$$

except for the only case $\{n = 4, s = 2, d = 2\}$.

Geometrically, the theorem says that one generic double line in $\mathbb{P}^n$ imposes independent conditions to hypersurfaces of given degree d containing s generic lines, with the exception of only the case $\{n = 4, s = 2, d = 2\}$.

A first generalization is asking not only for $m = 2$, but also for $m > 2$ arbitrary. Inspired by the question involving an arbitrary multiple line instead of a double line, and based on several examples, we conjecture that a similar result should hold, in analogy with the statement of Theorem 1.1. Namely we formulate the following conjecture:

> *The scheme $X \subset \mathbb{P}^n$, $n \geq 3$, consisting of $s \geq 1$ generic lines and one generic m-multiple line, $(m \geq 2)$, always has good postulation, except for the cases $\{n = 4, m = d, 2 \leq s \leq d\}$.*

From this conjecture we can deduce the important relation of the failure of X to have good postulation in degree d, with the multiplicity of a multiple line, the dimension of the ambient space, and the number of apparent simple lines. This seems to be fairly general behavior, which leads us to advance in more general situation. Indeed, one can push the problem we are facing even further, in the sense of substituting a multiple line with a multiple linear space of any dimension, and try to make a conjecture which parallels the above one.

Based on a similar analogy and some further evidence, we propose a conjecture as follows, which is significantly stronger than the former.

Conjecture 1.2 *Let $n, d, c \in \mathbb{N}$, and $n \geq c+2 \geq 3$. The scheme $X \subset \mathbb{P}^n$ consisting of $s \geq 1$ generic lines and one generic m-multiple linear space of dimension c, $(m \geq 2)$, always has good postulation, except for the cases*

$$\{n = c+3, m = d, 2 \leq s \leq d\}.$$

Of course, this conjecture coincides with the previous one for $c = 1$. As we shall see in Sect. 7, there are results that make the conjecture rather plausible. It is worth mentioning that the case $m = 1$ is considered in [3], where Ballico proved that a generic disjoint union of lines and one linear space always has good postulation [3, Proposition 1].

We want to finish by pointing out that a non-reduced scheme X supported on generic union of linear spaces always has exceptions, a phenomenon that does not happen when X is reduced (according to a conjecture in [7]). In fact, the "bad behavior" of X is always related to the multiple components of it.

The structure of the paper

Section 2 contains background material. To be more explicit, after recalling basic definitions and notations on the schemes of multiple linear spaces in Sect. 2.1, we then give, in Sect. 2.2, some lemmas and some elementary observations which are extremely useful in dealing with the postulation problem. Next, in Sect. 2.3 we collect the known results concerning the postulation of lines as well as of lines and one multiple point, that are necessary for our proofs in Sects. 4–6, in addition, we look at the Hilbert polynomial of multiple linear spaces. We will study the postulation of our schemes by a degeneration approach, namely, degeneration of two skew lines in such a way that the resulting degenerated scheme would be a sundial, in the sense of [8], thanks to a theorem of Carlini–Catalisano–Geramita on the postulation of sundials in any projective space; all this is represented in Sect. 2.4.

Section 3 making up the core of the paper devoted to the outline of the proof of our main theorem, Theorem 1.1. We begin this section with the exceptional case $\{n = 4, s = 2, d = 2\}$ of the theorem. We make explicit, in Sect. 3.1, the geometric reason that prevents a scheme consisting of one generic double line and two generic simple lines in $\mathbb{P}^4$ from imposing independent conditions to quadric hypersurfaces. Moreover, we solve completely the case of $d = 2$ of the theorem. In Sect. 3.2 we explain a rephrasing of Theorem 1.1, that is Theorem 3.2. So our goal will be to prove Theorem 1.1 in the reformulation of Theorem 3.2. Sect. 3.3 describes in detail our strategy for proving Theorem 3.2, which is based on geometric constructions of specialized and degenerated schemes, the well-known Horace lemma, and the intersection theory on a hyperplane or on a smooth quadric surface. We would like to point out that our method of degenerations owed to the works [8,10].

In order to apply the strategy we will use an induction procedure which has difficult but delicate beginning steps for $n = 3$ and $n = 4$. In Sects. 4 and 5 we prove Theorem 3.2 for, respectively, $n = 3$ and $n = 4$, setting the stage for our induction approach. While, the proof for the general case $n \geq 5$ will be carried out in Sect. 6.

Conjecture 1.2, which geometrically amounts to saying that one generic m-multiple c-dimensional linear space $m\mathbb{P}^c$ in $\mathbb{P}^n$, $(n \geq c + 2 \geq 3)$, fails to impose independent conditions to degree d hypersurfaces through s generic lines if and only if $n = c + 3$, $m = d$ and $2 \leq s \leq d$, is stated and discussed in Sect. 7, where we prove it for the exceptional cases, and we describe completely what happen for $d = m$.

Finally, in "Appendix", Sect. 8, we collect several numerical lemmas needed for our proofs in Sects. 5 and 6.

2 Background

We work throughout over an algebraically closed field k with characteristic zero.

2.1 Notations

Given a closed subscheme X of $\mathbb{P}^n$, $\mathcal{I}_X$ will denote the ideal sheaf of X. If X, Y are closed subschemes of $\mathbb{P}^n$ and $X \subset Y$, then we denote by $\mathcal{I}_{X,Y}$ the ideal sheaf of X in $\mathcal{O}_Y$.

If $\mathcal{F}$ is a coherent sheaf on the scheme X, for any integer $i \geq 0$ we use $h^i(X, \mathcal{F})$ to denote the k-vector space dimension of the cohomology group $H^i(X, \mathcal{F})$. In particular, when $X = \mathbb{P}^n$, we will often omit X and we will simply write $h^i(\mathcal{F})$.

A *(fat) point of multiplicity* m, or an m*-multiple point*, with support $P \in \mathbb{P}^n$, denoted mP, is the zero-dimensional subscheme of $\mathbb{P}^n$ defined by the ideal sheaf $(\mathcal{I}_P)^m$, i.e. the $(m-1)$th infinitesimal neighborhood of P. In case $P \in X$ for any smooth variety $X \subset \mathbb{P}^n$, we will write $mP|_X$ for the $(m-1)$th infinitesimal neighborhood of P in X, that is the schematic intersection of the m-multiple point mP of $\mathbb{P}^n$ and X with $(\mathcal{I}_{P,X})^m$ as its ideal sheaf.

Similarly, if $L \subset \mathbb{P}^n$ is a line (resp. linear space), the closed subscheme of $\mathbb{P}^n$ supported on L and defined by the ideal sheaf $(\mathcal{I}_L)^m$ is called a *(fat) line of multiplicity* m (resp. linear space), or an m*-multiple line* (resp. linear space), and is denoted by mL.

Let $m_1, \ldots, m_s$ be positive integers and let $X_1, \ldots, X_s$ be s closed subschemes of $\mathbb{P}^n$. We denote by

$$m_1 X_1 + \cdots + m_s X_s$$

the schematic union of $m_1 X_1, \ldots, m_s X_s$, i.e. the subscheme of $\mathbb{P}^n$ defined by the ideal sheaf $(\mathcal{I}_{X_1})^{m_1} \cap \cdots \cap (\mathcal{I}_{X_s})^{m_s}$.

2.2 Preliminary lemmas

The basic tool for the study of the postulation problem is the so called *Castelnuovo's inequality*, that is an immediate consequence of the well-known residual exact sequence (for more details see e.g. [2, Section 2]).

We first recall the notion of residual scheme [17, § 9.2.8].

Definition 2.1 Let X, Y be closed subschemes of $\mathbb{P}^n$.

(i) The closed subscheme of $\mathbb{P}^n$ defined by the ideal sheaf $(\mathcal{I}_X : \mathcal{I}_Y)$ is called the **residual** of X with respect to Y and denoted by $Res_Y(X)$.
(ii) The schematic intersection $X \cap Y$ defined by the ideal sheaf $(\mathcal{I}_X + \mathcal{I}_Y)/\mathcal{I}_Y$ of $\mathcal{O}_Y$ is called the **trace** of X on Y and denoted by $Tr_Y(X)$.

We note that the generally valid identity for ideal sheaves

$$(\mathcal{I}_{X_1} \cap \mathcal{I}_{X_2}) : \mathcal{I}_Y = (\mathcal{I}_{X_1} : \mathcal{I}_Y) \cap (\mathcal{I}_{X_2} : \mathcal{I}_Y)$$

implies that the residual of the schematic union $X_1 + X_2$ is the schematic union of the residuals.

Lemma 2.2 (Castelnuovo's Inequality) *Let $d, e \in \mathbb{N}$, and $d \geq e$. Let $H \subseteq \mathbb{P}^n$ be a hypersurface of degree e, and let $X \subseteq \mathbb{P}^n$ be a closed subscheme. Then*

$$h^0(\mathbb{P}^n, \mathcal{I}_X(d)) \leq h^0(\mathbb{P}^n, \mathcal{I}_{Res_H(X)}(d-e)) + h^0(H, \mathcal{I}_{Tr_H(X)}(d)).$$

This lemma, especially after the outstanding work of Hirschowitz [23], is the basis for a standard method of working inductively with degree to solve the postulation problem and particularly is central to our proofs in the present paper (Sects. 4–6).

The following remark is quite immediate.

Remark 2.3 Let $n, m, d, s, s' \in \mathbb{N}$, $s' < s$. Let L and $L_1, \ldots, L_s$ be generic lines in $\mathbb{P}^n$, $n \geq 3$. Let $X_s = mL + L_1 + \cdots + L_s \subset \mathbb{P}^n$.

(i) If $h^1(\mathcal{I}_{X_s}(d)) = 0$, then $h^1(\mathcal{I}_{X_{s'}}(d)) = 0$.
(ii) If $h^0(\mathcal{I}_{X_{s'}}(d)) = 0$, then $h^0(\mathcal{I}_{X_s}(d)) = 0$.

The following lemma shows that how to add a collection of collinear points to a scheme $X \subset \mathbb{P}^n$, in such a way that imposes independent conditions on the linear system of degree d hypersurfaces passing through X for a given degree d (which is a special case of [7, Lemma 2.2]).

Lemma 2.4 *Let $d \in \mathbb{N}$. Let $X \subseteq \mathbb{P}^n$ be a closed subscheme, and let $P_1, \ldots, P_s$ be generic points on a line $L \subset \mathbb{P}^n$.*
If $h^0(\mathcal{I}_X(d)) = s$ and $h^0(\mathcal{I}_{X+L}(d)) = 0$, then $h^0(\mathcal{I}_{X+P_1+\cdots+P_s}(d)) = 0$.

2.3 What results were previously known

As a key question in the direction of studying the postulation problem of a scheme $X \subset \mathbb{P}^n$ supported on generic union of linear spaces, one can ask: *What is the Hilbert polynomial of X?* When X is reduced, Derksen answered this question by giving a formula for computing the Hilbert polynomial of X (see [13] for details). Moreover, the Hilbert polynomial of a multiple linear space is well-known, and it is not difficult to verify it by a count of parameters, that can be found in e.g. [6, §2] and [15, Lemma 2.1].

Lemma 2.5 *Let $n, d, c \in \mathbb{N}$, $c < n$ and $1 \leq m \leq d$. Let $\Pi \subset \mathbb{P}^n$ be a linear space of dimension c, then*

$$HP(m\Pi, d) = \sum_{i=0}^{m-1} \binom{c+d-i}{c} \binom{n+i-c-1}{i}. \tag{1}$$

Indeed, the requirement for a degree d hypersurface in $\mathbb{P}^n$ to contain $m\Pi$, i.e. to have multiplicity m along the linear space Π, imposes the number of conditions on it, which is at most the right hand side of (1).

In our case, i.e. the case of double line, one knows that for a hypersurface to contain a double line $2L$ is equivalent to saying that it is singular along the line L, and the above lemma asserts that $2L$ in $\mathbb{P}^n$ imposes $(nd+1)$ independent conditions to degree d hypersurfaces.

Now we recall a few results on the postulation of schemes supported on generic linear spaces which we will use to prove our Theorem 1.1 in Sects. 4–6. We start with a spectacular result due to Hartshorne and Hirschowitz, about the generic lines.

Theorem 2.6 (Hartshorne–Hirschowitz) [22, Theorem 0.1] *Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let $\mathbb{X} \subset \mathbb{P}^n$ be a generic union of s lines. Then $\mathbb{X}$ has good postulation, i.e.,*

$$h^0(\mathcal{I}_{\mathbb{X}}(d)) = \max\left\{\binom{d+n}{n} - s(d+1), 0\right\}.$$

As a first step for positive dimensional non-reduced cases, in [1,10] and [5] the authors examined the postulation problem for a generic collection of skew lines and one fat point of multiplicity m in $\mathbb{P}^n$, $n \geq 3$, and they found out that when $n \geq 4$ these schemes have good postulation, but when $n = 3$ there are several defective such schemes. Now, one can present these results simultaneously in a theorem as follows (see [10, Theorem 3.2] for $n \geq 4$ and m arbitrary, [10, Proposition 4.1] for $n = 3$ and $m = 2$, [1, Theorem 1] for $n = 3$ and $m = 3$, and finally [5, Theorem 1] for $n = 3$ and $m \geq 4$).

Theorem 2.7 *Let $n, m, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one fat point of multiplicity $m \geq 2$. Then X has good postulation, i.e.,*

$$h^0(\mathcal{I}_X(d)) = \max\left\{\binom{d+n}{n} - \binom{m+n-1}{n} - s(d+1), 0\right\},$$

except for the cases $\{n = 3, m = d, 2 \leq s \leq d\}$.

Since we will apply the theorem for the case $m = 2$ and $d \geq 3$ several times in the next sections, it is convenient to restate it as follows.

Corollary 2.8 *Let $n, d \in \mathbb{N}$, and $n, d \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of $s \geq 1$ lines and one double point. Then X has good postulation in degree d, i.e.*

$$h^0(\mathcal{I}_X(d)) = \max\left\{\binom{d+n}{n} - (n+1) - s(d+1), 0\right\}.$$

2.4 A degeneration approach

A natural approach to the postulation problem is to argue by degeneration. In view of the fact that we have the semicontinuity theorem for cohomology groups in a flat family [21, III, 12.8], one may use the degenerations and the semicontinuity theorem in order to be able to better handle the postulation of schemes supported on generic unions of linear spaces. Specifically, if one can prove that the property of having good postulation is satisfied in the special fiber, i.e. the degenerate scheme, then one may hope to obtain the same property in the general fiber, i.e. the original scheme.

In the celebrated paper [22] Hartshorne and Hirschowitz investigated a new degeneration technique to attack the postulation problem for a generic union of lines. In fact, they degenerate two skew lines in $\mathbb{P}^3$ in such a way that the resulting scheme becomes a "degenerate conic with an embedded point" (which also was used in [23]). Even more generally, one can push this trick of "adding nilpotents" further, to give a degeneration of two skew lines in higher dimensional projective spaces $\mathbb{P}^n$, $n \geq 3$, this is what the authors introduced in [7, Definition 2.7 with $m = 1$] and called a *(3-dimensional) sundial*.

According to the terminology of [22], we say that C is a *degenerate conic* if C is the union of two intersecting lines L and M, so $C = L + M$.

Now we recall the definition of a 3-dimensional sundial or simply a sundial (see [8, Definition 3.7] or [7, Definition 2.7 with $m = 1$]).

Definition 2.9 Let L and M be two intersecting lines in $\mathbb{P}^n$, $n \geq 3$, and let $T \cong \mathbb{P}^3$ be a generic linear space containing the degenerate conic $L + M$. Let P be the singular point of $L + M$, i.e. $P = L \cap M$. We call the scheme $L + M + 2P|_T$ a *degenerate conic with an embedded point* or a *(3-dimensional) sundial*.

One can show a sundial is a flat limit inside $\mathbb{P}^n$ of a flat family whose general fiber is the disjoint union of two lines, i.e. a sundial is a degeneration of two generic lines in $\mathbb{P}^n$, $n \geq 3$. This is the content of the following lemma (see [22, Example 2.1.1] for the case $n = 3$, and [8, Lemma 3.8] or [7, Lemma 2.5 with $m = 1$] for the general case $n \geq 3$).

Lemma 2.10 *Let $X_1 \subset \mathbb{P}^n$, $n \geq 3$, be the disjoint union of two lines L_1 and M. Then there exists a flat family of subschemes $X_i \subset \langle X_1 \rangle \cong \mathbb{P}^3$, $(i \in k)$, whose general fiber is the union of two skew lines and whose special fiber is the sundial $X_0 = M + L + 2P|_{\langle X_1 \rangle}$, where L is a line and $M \cap L = P$.*

Note that we can also easily degenerate a simple generic point and a degenerate conic to a sundial. Therefore, a sundial is either a degeneration of two generic lines, or a degeneration of a scheme which is the union of a degenerate conic and a simple generic point [8, Remark 3.9].

We recall here the main result in [8] which guarantees that a generic collection of lines and sundials will behave well with respect to the postulation problem.

Theorem 2.11 [8, Theorem 4.4] *Let $n, d \in \mathbb{N}$, and $n \geq 3$. Let the scheme $X \subset \mathbb{P}^n$ be a generic union of x sundials and y lines. Then X has good postulation, i.e.*

$$h^0(\mathcal{I}_X(d)) = \max\left\{\binom{d+n}{n} - (2x+y)(d+1), 0\right\}.$$

3 Outline of the proof of Theorem 1.1

Now we have all the necessary tools to tackle our main theorem, Theorem 1.1.

3.1 The exceptional case

We look for the case where X fails to have good postulation. Actually, there is only one exception in this infinite family, namely the case $\{n = 4, s = 2\}$ which, $H^0(\mathcal{I}_X(2))$ has dimension one instead of zero. As we will see below, this exceptional case arises from geometric reason, although the proof follows from numerical reason.

Now we prove the following proposition, which completely describes the case $d = 2$ of Theorem 1.1:

Proposition 3.1 *The scheme $X \subset \mathbb{P}^n$, $n \geq 3$, consisting of $s \geq 1$ generic lines and one generic double line $2L$ has good postulation in degree $d = 2$, i.e.,*

$$h^0(\mathcal{I}_X(2)) = \max\left\{\binom{n+2}{n} - (2n+1) - 3s, 0\right\}$$
$$= \max\left\{\binom{n}{2} - 3s, 0\right\},$$

except for the only case $\{n = 4, s = 2\}$.

Proof The sections of $\mathcal{I}_X(2)$ correspond to quadric hypersurfaces in $\mathbb{P}^n$ which, in order to contain $2L$, have to be cones whose vertex contains the line L.

If $n = 3$, we obviously have $h^0(\mathcal{I}_X(2)) = 0$, as expected.

If $n \geq 4$, we consider the projection X' of X from L onto a generic linear subspace $\mathbb{P}^{n-2} \subset \mathbb{P}^n$, hence X' is a scheme consisting of s generic lines in $\mathbb{P}^{n-2}$. It follows that

$$h^0(\mathbb{P}^n, \mathcal{I}_X(2)) = h^0(\mathbb{P}^{n-2}, \mathcal{I}_{X'}(2)).$$

In case $n > 4$, by Hartshorne–Hirschowitz theorem (Theorem 2.6) we get

$$h^0(\mathbb{P}^{n-2}, \mathcal{I}_{X'}(2)) = \max\left\{\binom{2+n-2}{2} - 3s, 0\right\} = \max\left\{\binom{n}{2} - 3s, 0\right\},$$

and we get the conclusion.

In case $n = 4$, X' is a generic union of s lines in $\mathbb{P}^2$. Hence, for $s > 2$ it is immediate to see that $h^0(\mathcal{I}_{X'}(2)) = 0$. For $s \leq 2$ we have $h^0(\mathcal{I}_{X'}(2)) = \binom{2-s+2}{2} = \binom{4-s}{2}$, on the other hand the expected value for $h^0(\mathcal{I}_X(2))$ is $\max\{6 - 3s, 0\}$. Thus for $s = 1$, $h^0(\mathcal{I}_X(2)) = 3$, as expected; but for $s = 2$, $h^0(\mathcal{I}_X(2)) = 1$ while the expected one is 0, which is what we wanted to show. □

3.2 Rephrasing Theorem 1.1

From what we have observed in the previous subsection, it remains to verify Theorem 1.1 for $d \geq 3$, asserts that schemes $X \subset \mathbb{P}^n$ consisting of s generic lines and one generic double line have good postulation for all $d \geq 3$, i.e., $h^0(\mathcal{I}_X(d)) = 0$ or $h^1(\mathcal{I}_X(d)) = 0$. (Note that the case $d = 1$ of Theorem 1.1 being trivial, so we omit it.)

First note that, as X varies in a flat family, by the semicontinuity of cohomology [21, III, 12.8], the condition of good postulation, is clearly an open condition on the family of X. Hence to prove Theorem 1.1, it is enough to find any scheme of s lines and one double line, or even any scheme which is a specialization of a flat family of s lines and one double line, which has good postulation.

Given n and d, suppose one can choose s so that:

$$\binom{d+n}{n} = (nd+1) + s(d+1)$$

and suppose one can find X so that $h^0(\mathcal{I}_X(d)) = h^1(\mathcal{I}_X(d)) = 0$. Then if one removes some lines from X, one gets a scheme $X' \subset X$ such that $h^1(\mathcal{I}_{X'}(d))$ will still be zero (by Remark 2.3 (i)); and if one adds some lines to X, one gets a scheme $X'' \supset X$ such that $h^0(\mathcal{I}_{X''}(d))$ will still be zero (by Remark 2.3 (ii)). In other words, the good postulation for that given n, d and the unique integer s, that gives the equality above, implies the good postulation for the same n, d and any s whatsoever.

Unfortunately for given n, d one can not always find such an s. Therefore we will make adjustments by adding some collinear points to X, to get a similar equality. In particular, we prove the following theorem which, by Remark 2.3, implies our main Theorem 1.1 for $d \geq 3$:

Theorem 3.2 *Let $n, d \in \mathbb{N}$, and $n, d \geq 3$. Let*

$$r = \left\lfloor \frac{\binom{d+n}{n} - (nd+1)}{d+1} \right\rfloor; \quad q = \binom{d+n}{n} - (nd+1) - r(d+1).$$

Let the scheme $X \subset \mathbb{P}^n$ be the generic union of r lines $L_1, \ldots, L_r$, one double line $2L$ and q points $P_1, \ldots, P_q$ lying on a generic line M. Then X has good postulation, i.e.,

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \binom{d+n}{n} - (nd+1) - r(d+1) - q = 0.$$

From our discussion above, Theorem 1.1 follows immediately from this theorem. Indeed, to prove Theorem 1.1 for that n, d and any $r' \leq r$, simply remove the q points and $r - r'$ lines, then the corresponding $h^1(\mathcal{I}(d))$ will be zero; to prove it for $r'' > r$, first add the line M passing through the q collinear points, then add $r'' - r - 1$ disjoint lines, then the corresponding $h^0(\mathcal{I}(d))$ will be zero.

Remark 3.3 As we have seen, Theorem 3.2 implies Theorem 1.1. On the other hand, the converse also follows directly from Lemma 2.4.

3.3 Strategy of the proof

We illustrate our general strategy explicitly to prove Theorem 3.2 for a generic scheme $X \subset \mathbb{P}^n$ consisting of one double line, r simple lines and q collinear points, as follows:

The difficulty with proving a property like "good postulation" is that, it is very hard to lay hands on a *generic* scheme X. Our approach to overcome to this difficulty is to start with a special scheme, which is obtained by several different kind of specializations and degenerations, and then use semicontinuity theorem for cohomology groups [21, III, 12.8] to discover the same property for generic scheme X. The next step is to reduce the postulation problem of the specialized scheme, via Castelnuovo's inequality (Lemma 2.2), to the study of the postulation of a residual scheme and a trace scheme, that is *La méthode d'Horace*, elaborated by Hirschowitz [23].

To be more precise, for $n \geq 4$, we specialize x simple lines into a fixed hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ and degenerate q' other pairs of simple lines to sundials, further, we specialize these q' sundials into $\mathbb{P}^{n-1}$ unless their singular points. Thus if one can choose these numbers correctly, that is in such a way that the numbers x and q' are sufficiently many to comply with the induction hypothesis on degree (see "Appendix", Lemma 8.1), then the residual has good postulation, while the trace is a scheme in $\mathbb{P}^{n-1}$, which is more complicated to verify because of the appearance of q' degenerate conics and one double point. Now to handle the problem involving the postulation of the trace scheme we specialize $\bar{r}$ lines, $\bar{q}$ simple points, and the double point into a fixed hyperplane $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$, then we take again residual and trace. Of course, the numbers $\bar{r}$ and $\bar{q}$ should not be too numerous, and we have to find these numbers satisfying all the necessary inequalities (see "Appendix", Lemma 8.2). This time the trace consists of $\bar{r}$ lines, some simple points, and one double point, which by Corollary 2.8 has good postulation, while the residual consists of q' degenerate conics, $(x - \bar{r})$ lines and some simple points, which we will degenerate it to a scheme consisting of q' sundials, $(x - \bar{r})$ lines and some simple points, that by Theorem 2.11 has good postulation (these arrangements contain a lot of technical details which can be found in "Appendix", Lemma 8.2).

This argument for the trace of X can be applied in the case $n \geq 5$, but unfortunately does not cover the case $n = 4$, where forced intersection of lines appear in $\mathbb{P}^{n-2} = \mathbb{P}^2$. In fact the case $n = 4$ will be taken care of by a smooth quadric surface Q and a way of specialization which is considerably different from that mentioned above. Explicitly, we specialize one line of each of the degenerate conics, together with $\hat{r}$ simple lines, into the same ruling on Q, moreover, we specialize $\hat{q}$ simple points onto Q, then we take again residual and trace. Surely, the numbers $\hat{r}$ and $\hat{q}$ should not be so much, and we have to find these numbers satisfying all the necessary inequalities (see "Appendix", Lemma 8.3). Now the current residual consisting of one double point, $(x - \hat{r} + q')$ lines and some simple points will be verified by Corollary 2.8, while the current trace, which is a scheme in Q, will be verified by applying some results from internal geometry of Q.

What about in $\mathbb{P}^3$? Actually, the most difficult part of the proof is the case of $\mathbb{P}^3$. Our approach to this case uses extremely an ad hoc method which is done by specializing as many lines as is needed into a smooth quadric surface instead of a plane, and then, if necessary, by degenerating some other pairs of simple lines to sundials (and even more by specializing sundials and points if need be), that requires a case by case discussion. Here the role of the smooth quadric is explained by the property of having two rulings of skew lines and by the known results from intersection theory on it. We notice that also in the case of $\mathbb{P}^3$ our method can then be applied under certain numerical conditions, and this is why the proof splits into three specific cases $d \equiv 0 \pmod 3$, $d \equiv 1 \pmod 3$ and $d \equiv 2 \pmod 3$, which described exactly in Sect. 4. In fact, our method can be safely applied for $d \equiv 0 \pmod 3$, as well as for $d \equiv 1 \pmod 3$, but a slight complication arises in the case of $d \equiv 2 \pmod 3$, where we have to consider a different specializaton, which is done by placing the support of the double line into the smooth quadric.

Summing up, the method for proving our Theorem 3.2, based on the induction on degree d, breaks down into three parts: $n = 3$, $n = 4$, and $n \geq 5$, which we have to investigate each of them separately in Sects. 4–6.

We would like to point out that to make the strategy applicable, many verifications are needed because of the messy arithmetic involved (see Sect. 8).

Since to prove the property of good postulation, according to our strategy, we will use in the sequel Castelnuovo's inequality (Lemma 2.2) and the semicontinuity of cohomology ([21, III, 12.8]) several times, it will be useful to consider the following remark.

Remark 3.4 With the hypotheses of Theorem 3.2, let $\widetilde{X}$ be the scheme obtained from X by combining specializations and degenerations via a fixed hypersurface $H \subset \mathbb{P}^n$ of degree e.

If $h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-e)) = 0$ and $h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(d)) = 0$, then by Castelnuovo's inequality we have $h^0(\mathcal{I}_{\widetilde{X}}(d)) = 0$, and this implies, by the semicontinuity of cohomology, $h^0(\mathcal{I}_X(d)) = 0$.

4 Proof in $\mathbb{P}^3$

In this section we prove Theorem 3.2 in $\mathbb{P}^3$.

We start with a useful observation concerning the behavior of certain one-dimensional subschemes of a smooth quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ with respect to the linear system of curves of type (d, d) on Q, which we will often use in the sequel (for a proof see [22, Lemma 2.3]).

Lemma 4.1 *Let $\alpha, \beta, \gamma, \delta, d \in \mathbb{N}$, and let $Q \subset \mathbb{P}^3$ be a smooth quadric. Let the scheme $W \subset Q$ be a generic union of α lines belonging to the same ruling of Q, β simple points, γ simple points lying on a line belonging to the same ruling of the α lines, and δ double points. If the following conditions are satisfied:*

(1) $\alpha(d+1) + \beta + \gamma + 3\delta = (d+1)^2$;
(2) $\delta \leq d+1$;
(3) $\gamma \leq d+1$;
(4) *if* $d > \alpha$ *then* $\delta \leq \frac{d+1-\gamma}{2} + (d-\alpha-1)\left\lfloor \frac{d+1}{2} \right\rfloor$, *otherwise* $\delta = 0$;

then $h^1(Q, \mathcal{I}_W(d)) = h^0(Q, \mathcal{I}_W(d)) = 0$.

Before we begin our investigations in the case of $\mathbb{P}^3$, we recall some elementary facts about the geometry on the smooth quadric surface Q: the divisor class group of Q is $\mathbb{Z} \oplus \mathbb{Z}$, generated by a line in each of the two rulings; by the type we mean the class in $\mathbb{Z} \oplus \mathbb{Z}$; the curves on Q are those of type (a, b) with $a, b \geq 0$; by convention $\mathcal{O}_Q(d) = \mathcal{O}_Q(d, d)$; finally $h^0(Q, \mathcal{O}_Q(a, b)) = (a+1)(b+1)$.

Now we state and prove Theorem 3.2 in $\mathbb{P}^3$, that is:

Proposition 4.2 *Let $d \geq 3$ and*

$$r = \left\lfloor \frac{\binom{d+3}{3} - (3d+1)}{d+1} \right\rfloor; \quad q = \binom{d+3}{3} - (3d+1) - r(d+1).$$

Let the scheme $X \subset \mathbb{P}^3$ be the generic union of r lines $L_1, \ldots, L_r$, one double line $2L$ and q points $P_1, \ldots, P_q$ lying on a generic line M. Then X has good postulation, i.e.,

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \binom{d+3}{3} - (3d+1) - r(d+1) - q = 0.$$

Proof In order to start the induction argument we need to establish the base cases $d = 3, 4$.

First consider the case $d = 3$. In this case we have $r = 2$ and $q = 2$, therefore

$$X = 2L + L_1 + L_2 + P_1 + P_2 \subset \mathbb{P}^3.$$

Fix a generic plane $H \subset \mathbb{P}^3$, and consider the scheme $\widetilde{X}$ obtained from X by specializing the line L_1 and the points P_1, P_2 into H. By abuse of notation, we will again denote these specialized line and points by L_1 and P_1, P_2, respectively. (Keeping in mind that in the remainder of the paper, by abuse of notation, we will always denote the specialized components by the same letters as the original ones.)

We have

$$Res_H(\widetilde{X}) = 2L + L_2 \subset \mathbb{P}^3,$$

then it is obvious that

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(2)) = 0.$$

Also,

$$Tr_H(\widetilde{X}) = 2R|_H + L_1 + S + P_1 + P_2 \subset H \cong \mathbb{P}^2,$$

where $L \cap H = R$, so $2L \cap H = 2R|_H$ is a double point in H, and $L_2 \cap H = S$. Since L_1 is a fixed component for the sections of $\mathcal{I}_{Tr_H(\widetilde{X})}(3)$, we have

$$h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(3)) = h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})-L_1}(2)).$$

Since the double point $2R|_H$ imposes 3 independent conditions on $|\mathcal{O}_H(2)|$, and the points P_1, P_2, S are generic points in H, we easily get that

$$h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})-L_1}(2)) = \binom{2+2}{2} - 3 - 3 = 0.$$

Thus by Remark 3.4, with $(n = 3, e = 1, d = 3)$, we have $h^0(\mathcal{I}_X(3)) = 0$, that is, X has good postulation in degree 3.

Now consider the case $d = 4$. We have $r = 4$ and $q = 2$, then X is the schematic union:

$$X = 2L + L_1 + L_2 + L_3 + L_4 + P_1 + P_2 \subset \mathbb{P}^3.$$

Let Q be a smooth quadric surface, and let $\widetilde{X}$ be the scheme obtained from X by specializing three of the lines L_i in such a way that L_1, L_2, L_3 become lines of the same ruling on Q, and by specializing the points P_1, P_2 onto Q.

Then we get

$$Res_Q(\widetilde{X}) = 2L + L_4 \subset \mathbb{P}^3,$$

and it clearly follows that

$$h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(2)) = 0.$$

Consider the trace of $\widetilde{X}$ on Q, that is

$$Tr_Q(\widetilde{X}) = 2R_1|_Q + 2R_2|_Q + L_1 + L_2 + L_3 + S_1 + S_2 + P_1 + P_2 \subset Q,$$

where $L \cap Q = R_1 + R_2$, so $2L \cap Q = 2R_1|_Q + 2R_2|_Q$ consists of two double points on Q, and $L_4 \cap Q = S_1 + S_2$. Note that the scheme $Tr_Q(\widetilde{X})$ is generic union in Q of three lines belonging to the same ruling of Q, four simple points and two double points, hence we can apply Lemma 4.1, with $(\alpha = 3, \beta = 4, \gamma = 0, \delta = 2, d = 4)$, and we obtain

$$h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(4)) = 0.$$

Therefore by Remark 3.4, with $(n = 3, e = 2, d = 4)$, it follows that $h^0(\mathcal{I}_X(4)) = 0$. Hence the case $d = 4$ is done.

Now assume $d \geq 5$. We consider three cases, and we proceed by induction on d. Let Q be a smooth quadric surface in $\mathbb{P}^3$.

Case $d \equiv 0$ **(mod 3)** Write $d = 3t, t \geq 2$. Then

$$r = \frac{(t+1)(3t+2)}{2} - 3, \qquad q = 2.$$

We have

$$X = 2L + L_1 + \cdots + L_r + P_1 + P_2 \subset \mathbb{P}^3.$$

Since $2t+1 \leq r$, we specialize $2t+1$ of the lines L_i in such a way that $L_1, \ldots, L_{2t+1}$ become lines of the same ruling on Q, and we denote by $\widetilde{X}$ the specialized scheme. We have

$$Res_Q(\widetilde{X}) = 2L + L_{2t+2} + \cdots + L_r + P_1 + P_2 \subset \mathbb{P}^3,$$

which is the generic union of one double line, $\frac{t(3t+1)}{2} - 3$ lines and two points, so by the induction hypothesis it follows that

$$h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(d-2)) = 0.$$

Now we treat the trace scheme

$$\begin{aligned} Tr_Q(\widetilde{X}) = 2R_1|_Q + 2R_2|_Q + L_1 + \cdots + L_{2t+1} \\ + S_{1,2t+2} + S_{2,2t+2} + \cdots + S_{1,r} + S_{2,r} \subset Q, \end{aligned}$$

where $L \cap Q = R_1 + R_2$ and $L_i \cap Q = S_{1,i} + S_{2,i}$, $(2t+2 \leq i \leq r)$. Note that the points $R_1, R_2, S_{1,i}, S_{2,i}$, $(2t+2 \leq i \leq r)$, are generic points on Q. That is $Tr_Q(\widetilde{X})$ consists of $2t+1$ lines of the same ruling on Q, two generic double points and $t(3t+1) - 6$ generic simple points, then we can easily check that $Tr_Q(\widetilde{X})$ satisfies the conditions of Lemma 4.1, with $(\alpha = 2t+1, \beta = t(3t+1) - 6, \gamma = 0, \delta = 2)$, and this implies

$$h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(d)) = 0.$$

Hence by Remark 3.4, with $(n = 3, e = 2)$, we get $h^0(\mathcal{I}_X(d)) = 0$.

Case $d \equiv 1$ **(mod 3)** Write $d = 3t+1$, $t \geq 2$. Then

$$r = \frac{(t+1)(3t+4)}{2} - 3, \qquad q = 2.$$

In this case we have

$$X = 2L + L_1 + \cdots + L_r + P_1 + P_2 \subset \mathbb{P}^3.$$

We wish to construct a specialization of X so that the expected vanishing $h^0(\mathcal{I}_X(d)) = 0$ is obtained. In order to do this, we introduce the specialization $\widetilde{X}$ of X in the following way:

- specialize the points P_1, P_2 onto Q;
- specialize the first $2t+1$ lines L_i in such a way that they become lines of the same ruling on Q, and call the resulting set of lines X_1;

- degenerate the next $2t-2$ pairs of lines L_i, so that they become $2t-2$ sundials $\widehat{C_i} = C_i + 2N_i$, $(1 \le i \le 2t-2)$, where C_i is a degenerate conic and $2N_i$ is a double point with support at the singular point of C_i, furthermore, specialize the points $N_1, \dots, N_{2t-2}$ onto Q, and call the resulting scheme of sundials X_2, that is

$$X_2 = \widehat{C_1} + \cdots + \widehat{C_{2t-2}},$$

  with the property that the singular points of $\widehat{C_i}$ lie on Q;
- leave the remaining simple lines L_i, which are $r-(2t+1)-2(2t-2) = \frac{t(3t-5)}{2}+2$ lines, generic outside Q, and call this collection of lines X_3;

notice that we can do the above specialization because of the inequality $r \ge 2t+1+2(2t-2)$. Then by letting

$$\widetilde{X} = 2L + X_1 + X_2 + X_3 + P_1 + P_2 \subset \mathbb{P}^3,$$

we get the desired specialization of X.

Now we perform the process of verifying the residual and the trace on this specialized scheme $\widetilde{X}$. We obtain

$$\begin{aligned} Tr_Q(\widetilde{X}) = 2R_1|_Q + 2R_2|_Q + X_1 + Tr_Q(X_2) \\ + Tr_Q(X_3) + P_1 + P_2 \subset Q, \end{aligned}$$

where $L \cap Q = R_1 + R_2$; $Tr_Q(\widehat{C_i}) = C_i \cap Q + 2N_i|_Q$, and $C_i \cap Q$ is a union of two simple points, $(1 \le i \le 2t-2)$, therefore $Tr_Q(X_2)$ consists of $2t-2$ double points and $4t-4$ simple points; moreover, $Tr_Q(X_3)$ consists of $t(3t-5)+4$ simple points. Hence $Tr_Q(\widetilde{X})$ is generic union in Q of $2t+1$ lines belonging to the same ruling of Q, $2t$ double points and $t(3t-1)+2$ simple points. An easy computation, yields that the scheme $Tr_Q(\widetilde{X})$ verifies the conditions of Lemma 4.1, with $(\alpha = 2t+1, \beta = t(3t-1)+2, \gamma = 0, \delta = 2t)$, then we have

$$h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(d)) = 0.$$

So we are done with $Tr_Q(\widetilde{X})$. If we can prove $h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(d-2)) = 0$ then, by Castelnuovo's inequality, we get $h^0(\mathcal{I}_{\widetilde{X}}(d)) = 0$.

Here we consider the residual scheme

$$Res_Q(\widetilde{X}) = 2L + C_1 + \cdots + C_{2t-2} + X_3 \subset \mathbb{P}^3.$$

In order to compute $h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(d-2))$, we need to construct a specialization of $Res_Q(\widetilde{X})$, and take again the residual and the trace with respect to Q.

First, let $M_{1,i}$, $M_{2,i}$ be the two lines which form the degenerate conic C_i, $(1 \le i \le 2t-2)$, this means $C_i = M_{1,i} + M_{2,i}$. Pick a line $L' \subset X_3$. Now let $\widetilde{R}$ be the scheme obtained from $Res_Q(\widetilde{X})$ by specializing the degenerate conics C_i and the lines L, L' in such a way that the lines $M_{1,1}, \dots, M_{1,2t-2}$ and L, L' become $2t$ lines of the same ruling on Q (the lines $M_{2,1}, \ldots, M_{2,2t-2}$ and the other $\frac{t(3t-5)}{2}+1$ lines of X_3 remain generic lines, not lying on Q).

From this specialization we have

$$Res_Q(\widetilde{R}) = L + M_{2,1} + \cdots + M_{2,2t-2} + (X_3 - L') \subset \mathbb{P}^3,$$

that is generic union of $\frac{t(3t-1)}{2}$ lines in $\mathbb{P}^3$, hence by Hartshorne–Hirschowitz theorem (Theorem 2.6) we immediately get (note that $d = 3t+1$)

$$h^0(\mathcal{I}_{Res_Q(\widetilde{R})}(d-4)) = \binom{d-4+3}{3} - \frac{t(3t-1)}{2}(d-4+1) = 0.$$

On the other hand, $M_{2,i}$ meets Q in the two points which are $M_{1,i} \cap M_{2,i}$, that is contained in $M_{1,i}$, and another point, which we denote by S_i. Thus

$$\begin{aligned} Tr_Q(\widetilde{R}) = 2L|_Q + M_{1,1} + \cdots + M_{1,2t-2} + L' \\ + S_1 + \cdots + S_{2t-2} + Tr_Q(X_3 - L') \subset Q, \end{aligned}$$

where $Tr_Q(X_3 - L')$ is made by $t(3t-5)+2$ generic points. Therefore the scheme $Tr_Q(\widetilde{R})$ is generic union in Q of one double line $2L|_Q$, $2t-1$ simple lines, such that all of these $2t$ lines are placed in the same ruling of Q, and $3t(t-1)$ simple points. Considering Q as $\mathbb{P}^1 \times \mathbb{P}^1$ and assuming these lines belong to the first ruling of Q, we see that each of these simple lines is a curve of type $(1,0)$ and the double line $2L|_Q$ is a curve of type $(2,0)$.

Note that the double line $2L|_Q$ and the lines $M_{1,i}$, L', $(1 \leq i \leq 2t-2)$, are fixed components for the curves of $H^0(Q, \mathcal{I}_{Tr_Q(\widetilde{R})}(d-2, d-2))$, since $d-2 \geq 2t+1$. Now set

$$\Lambda = 2L|_Q + M_{1,1} + \cdots + M_{1,2t-2} + L' \subset Q,$$

which is of type $(2t+1, 0)$. Hence by removing the fixed component Λ, and by using the fact that the scheme $Tr_Q(\widetilde{R}) - \Lambda$ is generic union of $3t(t-1)$ simple points, moreover by recalling the equality $d = 3t+1$, we deduce

$$\begin{aligned} h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{R})}(d-2, d-2)) &= h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{R})-\Lambda}(d-2-(2t+1), d-2)) \\ &= h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{R})-\Lambda}(t-2, 3t-1)) \\ &= h^0(Q, \mathcal{O}_Q(t-2, 3t-1)) - 3t(t-1) \\ &= (t-1)3t - 3t(t-1) = 0. \end{aligned}$$

This together with $h^0(\mathcal{I}_{Res_Q(\widetilde{R})}(d-4)) = 0$, by Castelnuovo's inequality, implies that $h^0(\mathcal{I}_{\widetilde{R}}(d-2)) = 0$, and consequently, by semicontinuity,

$$h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(d-2)) = 0.$$

So we conclude that $h^0(\mathcal{I}_{\widetilde{X}}(d)) = 0$, and from here, again by semicontinuity, we get $h^0(\mathcal{I}_X(d)) = 0$.

Case $d \equiv 2$ **(mod 3)** Write $d = 3t + 2$, $t \geq 1$. Then

$$r = \frac{3t(t+3)}{2}, \qquad q = t + 3.$$

We have

$$X = 2L + L_1 + \cdots + L_r + P_1 + \cdots + P_{t+3} \subset \mathbb{P}^3,$$

where $P_1, \ldots, P_{t+3}$ are points lying on a generic line M.

Realize Q as $\mathbb{P}^1 \times \mathbb{P}^1$. We specialize $2t$ of the lines L_i and the lines L, M in such a way that $L_1, \ldots, L_{2t}$ and L, M become $2t + 2$ lines of the first ruling on Q, i.e. each has type $(1, 0)$, and we denote by $\widetilde{X}$ the specialized scheme (note that this is possible since $r \geq 2t + 2$).

First we consider the residual scheme

$$Res_Q(\widetilde{X}) = L + L_{2t+1} + \cdots + L_r \subset \mathbb{P}^3,$$

that is generic union of $r - 2t + 1 = \frac{(t+1)(3t+2)}{2}$ lines, so according to Hartshorne–Hirschowitz theorem (Theorem 2.6) we get (note that $d = 3t + 2$)

$$h^0(\mathcal{I}_{Res_Q(\widetilde{X})}(d-2)) = \binom{d-2+3}{3} - \frac{(t+1)(3t+2)}{2}(d-2+1) = 0.$$

Then we are left with the trace scheme, which is

$$Tr_Q(\widetilde{X}) = 2L|_Q + L_1 + \cdots + L_{2t} + X_1 + P_1 + \cdots + P_{t+3} \subset Q,$$

where $X_1 = Tr_Q(L_{2t+1} + \cdots + L_r)$. Using the fact that each L_i meets Q at two points, $(2t + 1 \leq i \leq r)$, it follows that X_1 is made by $2(r - 2t) = t(3t + 5)$ simple points.

Observe that the double line $2L|_Q$ and the lines $L_1, \ldots, L_{2t}$ are fixed components for the curves of $H^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(d, d))$, (note that $d \geq 2t + 2$). Set

$$\Lambda = 2L|_Q + L_1 + \cdots + L_{2t} \subset Q,$$

which has type $(2t + 2, 0)$. Removing the fixed component Λ implies that

$$\begin{aligned} h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(d, d)) &= h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})-\Lambda}(d - (2t+2), d)) \\ &= h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})-\Lambda}(t, 3t+2)). \end{aligned}$$

Hence we need to show that $h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})-\Lambda}(t, 3t+2)) = 0$, where

$$Tr_Q(\widetilde{X}) - \Lambda = X_1 + P_1 + \cdots + P_{t+3} \subset Q,$$

(recall that X_1 is a generic collection of $t(3t+5)$ points, and $P_1, \ldots, P_{t+3}$ are points on the line M belonging to the first ruling of Q). To see this, we wish to construct a specialization of $Tr_Q(\widetilde{X}) - \Lambda$ with the desired vanishing, we then must verify the residual and the trace in this new situation.

We start by choosing t lines $M_1, \ldots, M_t$ of the first ruling on Q, $M_i \neq M$. Next, let Y be the scheme obtained from $Tr_Q(\widetilde{X}) - \Lambda$ by specializing the $t(3t+3)$ points of X_1 onto the lines M_i in such a way that each of these lines contains exactly $3t+3$ of these points, and by specializing the remaining $2t$ points of X_1 onto the line M (this is possible because $t(3t+5) = 2t + t(3t+3)$).

Now suppose that C is a curve of $H^0(Q, \mathcal{I}_Y(t, 3t+2))$, i.e. a curve on Q of type $(t, 3t+2)$ containing Y. As we have just seen, the line M and also each line M_i, $(1 \leq i \leq t)$, contains $3t+3$ points of Y. The fact that C contains these points forces C to have the lines M, M_i as fixed components (since otherwise C must intersect M (resp. M_i) at $3t+2$ points, while C already passes through the $3t+3$ points of M (resp. M_i), which is impossible), but the number of these lines is $t+1$ and they are placed in the first ruling, which is a contradiction with the type $(t, 3t+2)$ of C. So such a C cannot exist, i.e., we have proved that $h^0(Q, \mathcal{I}_Y(t, 3t+2)) = 0$. Then by semicontinuity one can deduce that

$$h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})-\Lambda}(t, 3t+2)) = 0,$$

which is equivalent to

$$h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{X})}(d, d)) = 0.$$

Finally, from Remark 3.4, with $(n = 3, e = 2)$, we get the conclusion. □

5 Proof in $\mathbb{P}^4$

In this section we will prove Theorem 3.2 for the case $n = 4$.

Proposition 5.1 *Let $d \geq 3$ and*

$$r = \left\lfloor \frac{\binom{d+4}{4} - (4d+1)}{d+1} \right\rfloor; \quad q = \binom{d+4}{4} - (4d+1) - r(d+1).$$

Let the scheme $X \subset \mathbb{P}^4$ be the generic union of r lines $L_1, \ldots, L_r$, one double line $2L$ and q points $P_1, \ldots, P_q$ lying on a generic line M. Then X has good postulation, i.e.,

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \binom{d+4}{4} - (4d+1) - r(d+1) - q = 0.$$

Proof Let us begin with the case $d = 3$. In this case we have $r = 5$, and $q = 2$, then

$$X = 2L + L_1 + \cdots + L_5 + P_1 + P_2 \subset \mathbb{P}^4.$$

Pick a generic hyperplane $H \subset \mathbb{P}^4$. Now specialize the lines L, L_1 and also the points P_1, P_2 into H, and denote by $\widetilde{X}$ the specialized scheme.

On the one hand we obtain

$$Res_H(\widetilde{X}) = L + L_2 + \cdots + L_5 \subset \mathbb{P}^4,$$

that is, $Res_H(\widetilde{X})$ is union of 5 generic lines. Thus by Hartshorne–Hirschowitz theorem (Theorem 2.6) we immediately get

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(2)) = \binom{2+4}{4} - 15 = 0.$$

On the other hand we have

$$Tr_H(\widetilde{X}) = 2L|_H + L_1 + S_2 + \cdots + S_5 + P_1 + P_2 \subset H \cong \mathbb{P}^3,$$

where $L_i \cap H = S_i$, $(2 \le i \le 5)$. This means that $Tr_H(\widetilde{X})$ is generic union of one double line, one simple line, and 6 simple points in $H \cong \mathbb{P}^3$. Now Proposition 4.2, with $d = 3$, implies that Theorem 1.1 holds for the case $(n, d) = (3, 3)$. By applying Theorem 1.1, with $(n, d, s) = (3, 3, 1)$, we get that the scheme $2L|_H + L_1 \subset H \cong \mathbb{P}^3$ has good postulation in degree 3, i.e.,

$$h^0(H, \mathcal{I}_{2L|_H + L_1}(3)) = \binom{3+3}{3} - 10 - 4 = 6.$$

Since P_1, P_2 and S_i, $(2 \le i \le 5)$, are 6 generic points in H, we get

$$h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(3)) = 0.$$

Now by Remark 3.4, with $(n = 4, e = 1, d = 3)$, it follows that $h^0(\mathcal{I}_X(3)) = 0$.

Let us consider the case $d = 4$. Then $r = 10$ and $q = 3$. We observe that

$$X = 2L + L_1 + \cdots + L_{10} + P_1 + P_2 + P_3 \subset \mathbb{P}^4,$$

where P_1, P_2, P_3 are generic points lying on the line M.

Fix a generic hyperplane $H \subset \mathbb{P}^4$. Let $\widetilde{X}$ be the scheme obtained from X by specializing the lines L and L_1, L_2, L_3 into H.

We have

$$Res_H(\widetilde{X}) = X_1 + P_1 + P_2 + P_3 \subset \mathbb{P}^4,$$

where $X_1 = L + L_4 + \cdots + L_{10}$.

Applying Hartshorne–Hirschowitz theorem to X_1, which is union of 8 generic lines in $\mathbb{P}^4$, yields

$$h^0(\mathcal{I}_{X_1}(3)) = \binom{3+4}{4} - 32 = 3;$$

and also to $X_1 + M$, which is union of 9 generic lines in $\mathbb{P}^4$, yields

$$h^0(\mathcal{I}_{X_1+M}(3)) = \max\left\{\binom{3+4}{4} - 36, 0\right\} = 0.$$

Hence by Lemma 2.4 we get

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(3)) = 0.$$

Moreover, we have

$$Tr_H(\widetilde{X}) = 2L|_H + L_1 + L_2 + L_3 + S_4 + \cdots + S_{10} \subset H \cong \mathbb{P}^3,$$

where $L_i \cap H = S_i$, $(4 \leq i \leq 10)$.

By setting $X_2 = 2L|_H + L_1 + L_2 + L_3$, we see that X_2 is generic union in $H \cong \mathbb{P}^3$ of one double line and 3 simple lines, so by Theorem 1.1, with $(n, d, s) = (3, 4, 3)$, we obtain

$$h^0(H, \mathcal{I}_{X_2}(4)) = \binom{4+3}{3} - 13 - 15 = 7.$$

Notice that the points $S_4, \ldots, S_{10}$ are 7 generic points in H, therefore

$$h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(4)) = 0.$$

This together with $h^0(\mathcal{I}_{Res_H(\widetilde{X})}(3)) = 0$, by Castelnuovo's inequality, implies that $h^0(\mathcal{I}_{\widetilde{X}}(4)) = 0$, and from here, by semicontinuity, it follows the conclusion, which finishes the proof in this case.

Now assume $d \geq 5$. The rest of the proof will be by induction on d.

We start by letting

$$r' = \left\lfloor \frac{\binom{d+3}{4} - (4(d-1)+1) - q}{d} \right\rfloor;$$
$$q' = \binom{d+3}{4} - (4(d-1)+1) - r'd - q;$$
$$x = r - r' - 2q',$$

further, noting that $r', q', x \geq 0$ (see "Appendix", Lemma 8.1).

Recall that the scheme

$$X = 2L + L_1 + \cdots + L_r + P_1 + \cdots + P_q \subset \mathbb{P}^4,$$

is generic union of the double line $2L$, the r simple lines L_i, and the q points P_i belonging to the generic line M.

Fix a generic hyperplane $H \subset \mathbb{P}^4$. In order to prove that X has good postulation in degree d, we construct a scheme $\widetilde{X}$ obtained from X by combining specializations and degenerations as follows:

- specialize the first x lines L_i into H, and call the resulting set of lines X_1;
- degenerate the next q' pairs of lines L_i, so that they become q' sundials

  $$\widehat{C_i} = C_i + 2N_i|_{H_i}; \quad (1 \le i \le q'),$$

  where C_i is a degenerate conic, $H_i \cong \mathbb{P}^3$ is a generic linear space containing C_i and $2N_i|_{H_i}$ is a double point in H_i with support at the singular point of C_i, furthermore, specialize $\widehat{C_i}$ in such a way that $C_i \subset H$, but $2N_i|_{H_i} \not\subset H$, and call the resulting scheme of sundials X_2, that is

  $$X_2 = \widehat{C_1} + \cdots + \widehat{C_{q'}},$$

  with the property that the degenerate conics C_i lie in H, but $2N_i|_{H_i} \not\subset H$;
- leave the remaining simple lines L_i, which are $r' = r - x - 2q'$ lines, generic not lying in H, and call this collection of lines X_3;

then let

$$\widetilde{X} = 2L + X_1 + X_2 + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^4.$$

We need to show that $h^0(\mathcal{I}_{\widetilde{X}}(d)) = 0$, which clearly, by semicontinuity, implies that $h^0(\mathcal{I}_X(d)) = 0$. To do that, by Castelnuovo's inequality, it would be enough to show that $h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-1)) = 0$ and $h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(d)) = 0$.

First we verify the residual, which is

$$Res_H(\widetilde{X}) = 2L + Res_H(X_2) + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^4,$$

where $Res_H(X_2) = N_1 + \cdots + N_{q'}$. Recall that the points P_i are q generic points lying on the line M. In order to apply Lemma 2.4 to get $h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-1)) = 0$, it suffices to prove the two following equalities

$$h^0(\mathcal{I}_{2L+Res_H(X_2)+X_3}(d-1)) = q;$$
$$h^0(\mathcal{I}_{2L+Res_H(X_2)+X_3+M}(d-1)) = 0.$$

By the induction hypothesis we have that Proposition 5.1 holds with degree $d-1$, then Theorem 1.1 holds in $\mathbb{P}^4$ with degree $d-1$. Now by applying Theorem 1.1, with degree $d-1$, to the scheme $2L+X_3$, which consists of one double line and r' generic lines in $\mathbb{P}^4$, we get

$$h^0(\mathcal{I}_{2L+X_3}(d-1)) = \binom{d+3}{4} - (4(d-1)+1) - r'd$$
$$= q + q'.$$

Since $Res_H(X_2)$ consists of q' generic points, it immediately follows

$$h^0(\mathcal{I}_{2L+X_3+Res_H(X_2)}(d-1)) = q. \tag{2}$$

In the same way, by applying Theorem 1.1, with degree $d-1$, to the scheme $2L+X_3+M$, which consists of one double line and $r'+1$ generic lines in $\mathbb{P}^4$, we get

$$h^0(\mathcal{I}_{2L+X_3+M}(d-1)) = \max\left\{\binom{d+3}{4} - (4(d-1)+1) - (r'+1)d, 0\right\}$$
$$= \max\{q+q'-d, 0\},$$

and therefore (note that $q \leq d$ by the definition)

$$h^0(\mathcal{I}_{2L+X_3+M+Res_H(X_2)}(d-1)) = \max\{q-d, 0\} = 0. \tag{3}$$

Hence by (2) and (3) we have

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-1)) = 0,$$

so we are done with the residual scheme.

Now we treat the trace scheme $Tr_H(\widetilde{X})$, which we denote by T for short, that is

$$T = Tr_H(\widetilde{X}) = 2R|_H + X_1 + C_1 + \cdots + C_{q'} + X_3' \subset H \cong \mathbb{P}^3,$$

where $L \cap H = R$, thus $2L \cap H = 2R|_H$ is a double point in H, and $X_3' = Tr_H(X_3)$ is a generic collection of r' simple points; moreover, recall that X_1 is made by x generic lines, where $x = r - r' - 2q'$ as defined before.

We must prove that $h^0(H, \mathcal{I}_T(d)) = 0$. In order to do this, we wish to construct a specialization of T, with the desired vanishing, but this time our specialization will be via a smooth quadric surface in H. Since our investigations of T will be done in $H \cong \mathbb{P}^3$, as the ambient space, then for simplicity of notation we will from now on write $\mathbb{P}^3$ instead of H, as well as, $2R$ instead of $2R|_H$.

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric in $\mathbb{P}^3$. Notations and terminology concerning Q are those of Sect. 4. Let

$$\hat{r} = \left\lfloor \frac{(d+1)^2 - (d+2)q' - 2x}{d-1} \right\rfloor,$$
$$\hat{q} = (d+1)^2 - (d+2)q' - (d-1)\hat{r} - 2x.$$

Note that $\hat{r} \geq 0$, and so $\hat{q} \geq 0$ (see "Appendix", Lemma 8.3 (i)).

To begin, let $M_{1,i}, M_{2,i}$ be the two lines which form the degenerate conic C_i, $(1 \leq i \leq q')$, this means $C_i = M_{1,i} + M_{2,i}$, and let $S_1, \ldots, S_{r'}$ be the points of X'_3. Because of the inequalities $\hat{r} \leq x$ and $\hat{q} \leq r'$, (both are proved in "Appendix", Lemma 8.3 (ii), (iii)), we can specialize T in the following way.

Let $\widetilde{T}$ be the scheme obtained from T by specializing the degenerate conics C_i and $\hat{r}$ lines $L_1, \ldots, L_{\hat{r}}$ of X_1 in such a way that the lines $M_{1,1}, \ldots, M_{1,q'}$ and $L_1, \ldots, L_{\hat{r}}$ become lines belonging to the first ruling of Q, and by specializing $\hat{q}$ points $S_1, \ldots, S_{\hat{q}}$ of X'_3 onto Q (the lines $M_{2,1}, \ldots, M_{2,q'}$ and the other lines $L_{\hat{r}+1}, \ldots, L_x$ of X_1, also the remaining points $S_{\hat{q}+1}, \ldots, S_{r'}$ of X'_3 and the point R, remain generic not lying on Q).

Next, we perform the process of treating the residual and the trace of the specialized scheme $\widetilde{T}$, with respect to Q, to get $h^0(\mathbb{P}^3, \mathcal{I}_{\widetilde{T}}(d)) = 0$.

We have

$$Res_Q(\widetilde{T}) = 2R + L_{\hat{r}+1} + \cdots + L_x + M_{2,1} + \cdots + M_{2,q'} + S_{\hat{q}+1} + \cdots + S_{r'} \subset \mathbb{P}^3.$$

Observe that the scheme $Res_Q(\widetilde{T}) - (S_{\hat{q}+1} + \cdots + S_{r'})$ is generic union of one double point and $q' + x - \hat{r}$ lines in $\mathbb{P}^3$, and that $d - 2 \geq 3$, thus by Corollary 2.8 we get

$$h^0(\mathcal{I}_{Res_Q(\widetilde{T})-(S_{\hat{q}+1}+\cdots+S_{r'})}(d-2)) = \binom{d-2+3}{3} - 4 - (q' + x - \hat{r})(d-1) = r' - \hat{q},$$

(the last equality is proved in "Appendix", Lemma 8.3 (v)). Moreover, the points $S_{\hat{q}+1}, \ldots, S_{r'}$ are $r' - \hat{q}$ generic points, so we immediately get

$$h^0(\mathcal{I}_{Res_Q(\widetilde{T})}(d-2)) = 0.$$

Now it remains to consider the trace scheme. We first notice that $M_{2,i}$ meets Q in the two points which are $(M_{1,i} \cap M_{2,i})$ and another point, which we denote by S'_i, also recall that $M_{1,i} \subset Q$, so we have that $C_i \cap Q = M_{1,i} + S'_i$, $(1 \leq i \leq q')$. Similarly, L_j, $(\hat{r}+1 \leq j \leq x)$, meets Q in two points, then $Tr_Q(L_{\hat{r}+1} + \cdots + L_x)$ is a collection of $2(x - \hat{r})$ points, which we denote by T_1. Thus we obtain

$$Tr_Q(\widetilde{T}) = L_1 + \cdots + L_{\hat{r}} + T_1 + M_{1,1} + \cdots + M_{1,q'} + S'_1 + \cdots + S'_{q'} + S_1 + \cdots + S_{\hat{q}} \subset Q.$$

Since the lines $M_{1,i}$ and L_j, $(1 \leq i \leq q';\ 1 \leq j \leq \hat{r})$, are contained in the first ruling of Q, furthermore $d \geq q' + \hat{r}$ (see "Appendix", Lemma 8.3 (iv)), then all of these lines are fixed components for the curves of $H^0(Q, \mathcal{I}_{Tr_Q(\widetilde{T})}(d,d))$. Set

$$\Lambda = L_1 + \cdots + L_{\hat{r}} + M_{1,1} + \cdots + M_{1,q'} \subset Q,$$

which is of type $(q' + \hat{r}, 0)$. Now by removing the fixed component Λ, and by using the fact that the points S_i', S_k, $(1 \le i \le q';\ 1 \le k \le \hat{q})$, are generic on Q, as well as the points of T_1, we conclude that

$$\begin{aligned} h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{T})}(d,d)) &= h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{T})-\Lambda}(d - q' - \hat{r}, d)) \\ &= h^0(Q, \mathcal{O}_Q(d - q' - \hat{r}, d)) - (q' + \hat{q} + 2x - 2\hat{r}) \\ &= (d - q' - \hat{r} + 1)(d+1) - (q' + \hat{q} + 2x - 2\hat{r}) \\ &= (d+1)^2 - (d+2)q' - (d-1)\hat{r} - 2x - \hat{q} = 0. \end{aligned}$$

Putting together $h^0(\mathcal{I}_{Res_Q(\widetilde{T})}(d-2)) = 0$ and $h^0(Q, \mathcal{I}_{Tr_Q(\widetilde{T})}(d,d)) = 0$, from Castelnuovo's inequality we have $h^0(\mathcal{I}_{\widetilde{T}}(d)) = 0$, therefore, by semicontinuity, we have $h^0(\mathcal{I}_T(d)) = 0$. This completes the proof. □

6 Proof in $\mathbb{P}^n$ for $n \ge 5$

We come to the general case $n \ge 5$. Now we have the bases for our inductive approach, we are ready to prove Theorem 3.2 in the general setting.

Proposition 6.1 *Let $n, d \in \mathbb{N}$, and $n \ge 5$, $d \ge 3$. Let*

$$r = \left\lfloor \frac{\binom{d+n}{n} - (nd+1)}{d+1} \right\rfloor; \quad q = \binom{d+n}{n} - (nd+1) - r(d+1).$$

Let the scheme $X \subset \mathbb{P}^n$ be the generic union of r lines $L_1, \dots, L_r$, one double line $2L$ and q points $P_1, \dots, P_q$ lying on a generic line M. Then X has good postulation, i.e.,

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) = \binom{d+n}{n} - (nd+1) - r(d+1) - q = 0.$$

Proof We will prove the theorem by induction on d. We proceed to the general case of $n \ge 5$, noting that Theorem 3.2 have been proved for the cases $\mathbb{P}^3$ and $\mathbb{P}^4$ in Propositions 4.2 and 5.1.

To begin, let

$$\begin{aligned} r' &= \left\lfloor \frac{\binom{d-1+n}{n} - (n(d-1)+1) - q}{d} \right\rfloor; \\ q' &= \binom{d-1+n}{n} - (n(d-1)+1) - r'd - q; \\ x &= r - r' - 2q', \end{aligned}$$

we can check that $r', q', x \ge 0$ (see "Appendix", Lemma 8.1).

Let $H \subset \mathbb{P}^n$ be a generic hyperplane. For the purpose of getting $h^0(\mathcal{I}_X(d)) = 0$, we wish to find a scheme $\widetilde{X}$ obtained from X by combining specializations and degenerations so that the desired vanishing can be achieved. Now we construct the required $\widetilde{X}$ in the following way, which is analogous to the one used in $\mathbb{P}^4$ in the previous section:

- specialize the first x lines L_i into H, and call the resulting set of lines X_1;
- degenerate the next q' pairs of lines L_i, so that they become q' sundials

  $$\widehat{C_i} = C_i + 2N_i|_{H_i}; \quad (1 \leq i \leq q'),$$

  where C_i is a degenerate conic, $H_i \cong \mathbb{P}^3$ is a generic linear space containing C_i and $2N_i|_{H_i}$ is a double point in H_i with support at the singular point of C_i, furthermore, specialize $\widehat{C_i}$ in such a way that $C_i \subset H$, but $2N_i|_{H_i} \not\subset H$, and call the resulting scheme of sundials X_2, that is

  $$X_2 = \widehat{C_1} + \cdots + \widehat{C_{q'}};$$

- leave the remaining simple lines L_i, which are $r' = r - x - 2q'$ lines, generic not lying in H, and call this collection of lines X_3;

then let

$$\widetilde{X} = 2L + X_1 + X_2 + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^n.$$

To show that $h^0(\mathcal{I}_{\widetilde{X}}(d)) = 0$, by Castelnuovo's inequality, our goal will be to show that the following vanishings

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-1)) = 0; \quad h^0(H, \mathcal{I}_{Tr_H(\widetilde{X})}(d)) = 0.$$

With regard to residual, we have

$$Res_H(\widetilde{X}) = 2L + Res_H(X_2) + X_3 + P_1 + \cdots + P_q \subset \mathbb{P}^n,$$

where $Res_H(X_2) = N_1 + \cdots + N_{q'}$.

By the induction hypothesis we know that Proposition 6.1 holds with degree $d-1$, which implies that Theorem 1.1 holds with degree $d-1$ too (note that $n \geq 5$, then, even if $d-1=2$, by Proposition 3.1 we have that Theorem 1.1 holds for $d=2$). So we can apply Theorem 1.1, with degree $d-1$, to the scheme $2L + X_3$, as well as, to the scheme $2L + X_3 + M$, therefore

$$\begin{aligned} h^0(\mathcal{I}_{2L+X_3}(d-1)) &= \binom{d-1+n}{n} - (n(d-1)+1) - r'd \\ &= q + q'; \end{aligned}$$

$$h^0(\mathcal{I}_{2L+X_3+M}(d-1)) = \max\left\{\binom{d-1+n}{n} - (n(d-1)+1) - (r'+1)d, 0\right\}$$
$$= \max\{q+q'-d, 0\}.$$

Observe that $Res_H(X_2)$ is made by q' generic points, so we get

$$h^0(\mathcal{I}_{2L+X_3+Res_H(X_2)}(d-1)) = q; \tag{4}$$
$$h^0(\mathcal{I}_{2L+X_3+M+Res_H(X_2)}(d-1)) = \max\{q-d, 0\} = 0. \tag{5}$$

Having (4) and (5), moreover, recalling that the points P_i are q generic points lying on the line M, we can now apply Lemma 2.4, hence

$$h^0(\mathcal{I}_{Res_H(\widetilde{X})}(d-1)) = 0,$$

as we wanted.

Now, we consider the trace scheme $Tr_H(\widetilde{X})$, which we denote by T for short,

$$T = Tr_H(\widetilde{X}) = 2R|_H + X_1 + C_1 + \cdots + C_{q'} + X_3' \subset H \cong \mathbb{P}^{n-1},$$

where $L \cap H = R$, thus $2L \cap H = 2R|_H$ is a double point in H, and $X_3' = Tr_H(X_3)$ is a generic collection of r' simple points, which we denote by $S_1, \ldots, S_{r'}$. In addition, recall that X_1 is made by x generic lines, where $x = r - r' - 2q'$ as defined before.

For simplicity in the notation, we will henceforward write $\mathbb{P}^{n-1}$ instead of H, as well as, $2R$ instead of $2R|_H$.

In order to verify the scheme T, we make a specialization $\widetilde{T}$ of T via a fixed hyperplane as follows: we start by setting

$$\bar{r} = \left\lfloor \frac{\binom{d+n-2}{n-2} - (n-1) - r + r'}{d} \right\rfloor;$$
$$\bar{q} = \binom{d+n-2}{n-2} - (n-1) - \bar{r}d - r + r',$$

also noting that $\bar{r}, \bar{q} \geq 0$ ("Appendix", Lemma 8.2 (i)). Pick a generic hyperplane $H' \cong \mathbb{P}^{n-2}$ in $\mathbb{P}^{n-1}$. Now using the inequalities $\bar{r} \leq x$ and $\bar{q} \leq r'$, (both are proved in "Appendix", Lemma 8.2 (ii), (iii)), we specialize the lines $L_1, \ldots, L_{\bar{r}}$ of X_1, also the points $S_1, \ldots, S_{\bar{q}}$ of X_3' and the point R into H', and we denote by $\widetilde{T}$ the specialized scheme (note that the other lines $L_{\bar{r}+1}, \ldots, L_x$ of X_1, the degenerate conics C_i, and the other points of X_3' remain generic outside H').

Now in order to prove that $h^0(\mathbb{P}^{n-1}, \mathcal{I}_T(d)) = 0$, by semicontinuity, our next goal will be to prove that $h^0(\mathbb{P}^{n-1}, \mathcal{I}_{\widetilde{T}}(d)) = 0$.

L_i meets H' at one point, $(\bar{r}+1 \leq i \leq x)$, so $Tr_{H'}(L_{\bar{r}+1} + \cdots + L_x)$ is a union of $x - \bar{r}$ points, which we denote by T_1. Moreover, C_j meets H' in two points, $(1 \leq j \leq q')$, then $Tr_{H'}(C_1 + \cdots + C_{q'})$ is a collection of $2q'$ points, which we denote by T_2. Accordingly with these notations, we have

$$Tr_{H'}(\widetilde{T}) = 2R|_{H'} + L_1 + \cdots + L_{\bar{r}} + T_1 + T_2 \\ +S_1 + \cdots + S_{\bar{q}} \subset H' \cong \mathbb{P}^{n-2}.$$

First we apply Corollary 2.8 to the scheme $2R|_{H'} + L_1 + \cdots + L_{\bar{r}}$, which implies that

$$\begin{aligned} h^0(H', \mathcal{I}_{2R|_{H'}+L_1+\cdots+L_{\bar{r}}}(d)) &= \binom{d+n-2}{n-2} - (n-1) - \bar{r}(d+1) \\ &= \bar{q} - \bar{r} + r - r' \\ &= \bar{q} - \bar{r} + 2q' + x, \end{aligned}$$

next, by the fact that the schematic union $(T_1 + T_2 + S_1 + \cdots + S_{\bar{q}})$ is a generic union of $x - \bar{r} + 2q' + \bar{q}$ simple points, we immediately get

$$h^0(H', \mathcal{I}_{Tr_{H'}(\widetilde{T})}(d)) = 0, \tag{6}$$

so we are finished with the trace scheme.

Then we are left with the residual of $\widetilde{T}$ with respect to $H' \cong \mathbb{P}^{n-2}$, which is

$$Res_{H'}(\widetilde{T}) = R + C_1 + \cdots + C_{q'} + L_{\bar{r}+1} + \cdots + L_x \\ +S_{\bar{q}+1} + \cdots + S_{r'} \subset \mathbb{P}^{n-1}.$$

It is the existence of the degenerate conics C_i that impedes us to directly investigate the residual scheme. Our method to afford this difficulty is to take a degeneration of $Res_{H'}(\widetilde{T})$, but using a different way to do so. Indeed, according to the observation of Sect. 2.4 saying that a sundial can be considered as a degeneration of a degenerate conic together with a simple point, we then degenerate q' points $S_{\bar{q}+1}, \ldots, S_{\bar{q}+q'}$ together with q' conics C_i so that they become q' sundials $\widehat{C}_i$, (it is possible because $\bar{q} + q' \le r'$, "Appendix", Lemma 8.2 (iii)). We set $\Gamma = R + S_{\bar{q}+q'+1} + \cdots + S_{r'}$.

Let Y be the scheme obtained from $Res_{H'}(\widetilde{T})$ by this degeneration, more precisely,

$$Y = \widehat{C_1} + \cdots + \widehat{C_{q'}} + L_{\bar{r}+1} + \cdots + L_x + \Gamma \subset \mathbb{P}^{n-1}.$$

The scheme $Y - \Gamma$ is generic union of q' sundials and $x - \bar{r}$ lines in $\mathbb{P}^{n-1}$, so by Theorem 2.11, has good postulation, in other words

$$\begin{aligned} h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Y-\Gamma}(d-1)) &= \binom{d-1+n-1}{n-1} - (2q' + x - \bar{r})d \\ &= r' - q' - \bar{q} + 1, \end{aligned}$$

the computations to get the last equality can be found in "Appendix", Lemma 8.2 (iv). Since Γ is generic union of $r' - \bar{q} - q' + 1$ points, it then immediately follows that

$$h^0(\mathbb{P}^{n-1}, \mathcal{I}_Y(d-1)) = 0,$$

and from here, again by semicontinuity, we obtain

$$h^0(\mathbb{P}^{n-1}, \mathcal{I}_{Res_{H'}(\widetilde{T})}(d-1)) = 0.$$

This together with (6), by Castelnuovo's inequality, yields that

$$h^0(\mathbb{P}^{n-1}, \mathcal{I}_{\widetilde{T}}(d)) = 0,$$

and this is in fact what we wanted to show, hence the proof is complete. □

7 On Conjecture 1.2

Now coming back to our Conjecture 1.2, we will prove it only in a special case.

7.1 Some evidence for Conjecture 1.2

The main result of this paper, Theorem 1.1, attracts our attention to a natural class of objects that is schemes X of lines and one fat linear space in projective space. In fact, the geometry of the exception that we determined in Theorem 1.1 (see Proposition 3.1) leads us to conjecture that it can be generalized somehow to the families of lines and one fat linear space. The basic motivation lies in the fact that, no defective cases with respect to the linear system $|\mathcal{I}_X(d)|$ have been discovered, unless $d = m$, where m is the multiplicity of that linear space. So we hope the following conjecture, which exactly describes the failure of X to have good postulation.

Conjecture 7.1 (Conjecture 1.2 of the Introduction) *Let $n, d, c \in \mathbb{N}$, and $n \geq c+2 \geq 3$. The scheme $X \subset \mathbb{P}^n$ consisting of $s \geq 1$ generic lines and one generic m-multiple linear space $m\Pi$, $(m \geq 2)$, with $\Pi \cong \mathbb{P}^c \subset \mathbb{P}^n$, always has good postulation, except for the cases*

$$\{n = c+3, m = d, 2 \leq s \leq d\}.$$

This conjecture would be in perfect analogy with Theorem 1.1. Note that it is a hard problem to prove it in general case, and doing so requires the most sophisticated investigations with a lot of technical details, in the setting of specialization and degeneration.

Now we show that the conjecture is true for the special case of $d = m$, which is in the center of our attention. Before proceeding to state and prove it, let us introduce the following integer $\alpha_{(n,d;c,m)}$, for all integers n, c, d, m with $n > c$ and $d \geq m-1$, which we will use throughout this section:

$$\alpha_{(n,d;c,m)} = \sum_{i=0}^{m-1} \binom{c+d-i}{c}\binom{n+i-c-1}{i}.$$

Observe that $\alpha_{(n,d;c,m)}$ is exactly the Hilbert polynomial of $m\Pi$ in degree d, Lemma 2.5. Moreover, when $d = m$ with a straightforward computation, one easily sees that:

$$\alpha_{(n,m;c,m)} = \binom{n+m}{n} - \binom{n+m-c-1}{n-c-1}.$$

Proposition 7.2 *The scheme $X \subset \mathbb{P}^n, n \geq c+2 \geq 3$, consisting of $s \geq 1$ generic lines and one generic m-multiple linear space $m\Pi$, $(m \geq 2)$, with $\Pi \cong \mathbb{P}^c \subset \mathbb{P}^n$, has good postulation in degree m, i.e.,*

$$\begin{aligned} h^0(\mathcal{I}_X(m)) &= \max\left\{\binom{m+n}{n} - \alpha_{(n,m;c,m)} - s(m+1), 0\right\} \\ &= \max\left\{\binom{n+m-c-1}{n-c-1} - s(m+1), 0\right\}, \end{aligned}$$

except for $\{n = c+3, 2 \leq s \leq m\}$, in which case the defect is $\binom{s}{2}$.

Proof First notice that, the sections of $\mathcal{I}_X(m)$ correspond to degree m hypersurfaces in $\mathbb{P}^n$ which, in order to contain $m\Pi$, have to be cones whose vertex contains the linear space Π.

For $n = c+2$, it is easy to see that the linear system $|\mathcal{I}_X(m)|$ is empty, i.e. $h^0(\mathcal{I}_X(m)) = 0$, that is what was expected.

For $n \geq c+3$, let us consider the projection X' of X from Π into a generic linear subspace $\mathbb{P}^{n-c-1} \subset \mathbb{P}^n$. Then we have that the scheme X' consists of s generic lines in $\mathbb{P}^{n-c-1}$, also that the following equality

$$h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = h^0(\mathbb{P}^{n-c-1}, \mathcal{I}_{X'}(m)).$$

In case $n > c+3$, we have $n-c-1 \geq 3$, therefore from Hartshorne–Hirschowitz theorem (Theorem 2.6) it follows that

$$h^0(\mathbb{P}^{n-c-1}, \mathcal{I}_{X'}(m)) = \max\left\{\binom{m+n-c-1}{n-c-1} - s(m+1), 0\right\},$$

which is the expected value for $h^0(\mathbb{P}^n, \mathcal{I}_X(m))$, so we are done in this case.

In case $n = c+3$, X' is a generic union of s lines in $\mathbb{P}^2$. Hence, if $s > m$, we obviously have $h^0(\mathbb{P}^2, \mathcal{I}_{X'}(m)) = 0$, as expected. If $s \leq m$, we have $h^0(\mathbb{P}^2, \mathcal{I}_{X'}(m)) = \binom{m-s+2}{2}$, and consequently $h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = \binom{m-s+2}{2}$, on the other hand the expected value for $h^0(\mathbb{P}^n, \mathcal{I}_X(m))$ is

$$\max\left\{\binom{m+2}{2} - s(m+1), 0\right\},$$

which we denote by $\exp h^0(\mathbb{P}^n, \mathcal{I}_X(m))$. Thus for $s = 1$, we get that $h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = \binom{m+1}{2}$, as expected; but for $2 \le s \le m$, we get that

$$h^0(\mathbb{P}^n, \mathcal{I}_X(m)) \neq \exp h^0(\mathbb{P}^n, \mathcal{I}_X(m))$$

and the defect is

$$h^0(\mathbb{P}^n, \mathcal{I}_X(m)) - \exp h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = \binom{s}{2},$$

which finishes the proof. □

7.2 Final remark

A complete proof for Conjecture 7.1 (Conjecture 1.2 of the Introduction) will be a substantial effort, however, we believe that a method analogous to that presented in Sect. 3.3, can be successfully applied for studying postulation problem for a generic scheme of lines and one fat linear space in $\mathbb{P}^n$, and we plan to study this problem in the future. Indeed, if one can provide a proof for Conjecture 7.1 in the case of a generic union of lines and one fat line in $\mathbb{P}^n$, then even interestingly enough, one may hope to generalize this approach to the case of a generic union of lines and one fat linear space, that seems to be quite difficult. Actually, compared with the proof we gave in this paper, in the case of lines and one fat linear space we are forced to divide the proof in much more steps. While an argument analogous to Theorem 1.1 works in a more complicated way for the higher dimensional ambient projective spaces, investigations in two initial ambient spaces cause troubles, and this is why we leave it for the future.

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8 Appendix: Calculations

Lemma 8.1 *Let $n \ge 5, d \ge 3$ or $n = 4, d \ge 5$. Let*

$$r = \left\lfloor \frac{\binom{d+n}{n} - (nd+1)}{d+1} \right\rfloor, \quad q = \binom{d+n}{n} - (nd+1) - r(d+1);$$

$$r' = \left\lfloor \frac{\binom{d-1+n}{n} - (n(d-1)+1) - q}{d} \right\rfloor,$$

$$q' = \binom{d-1+n}{n} - (n(d-1)+1) - r'd - q.$$

Then

(i) $r' \geq 0$;
(ii) $r - r' - 2q' \geq 0$.

Proof (i) Since $q \leq d$, we have

$$\binom{d-1+n}{n} - (n(d-1)+1) - q \geq \binom{d-1+n}{n} - (n(d-1)+1) - d,$$

so in order to show that $r' \geq 0$ it is enough to show that

$$\binom{d-1+n}{n} - (n(d-1)+1) \geq d. \tag{7}$$

First consider the case $n = 4$ and $d \geq 5$, then we obviously have

$$\binom{d-1+n}{n} - (n(d-1)+1) - d = \binom{d+3}{4} - 5d + 3 \geq 0.$$

Now consider the case $n \geq 5$ and $d \geq 3$. Notice that the function $\binom{d-1+n}{n} - (n(d-1)+1)$ is an increasing function in n, hence to get the conclusion it suffices to prove the inequality (7) only for $n = 5$. Now by letting $n = 5$, we easily see that

$$\begin{aligned}\binom{d-1+n}{n} - (n(d-1)+1) - d &= \binom{d-1+5}{5} - (5(d-1)+1) - d \\ &= \binom{d+4}{5} - 6d + 4 \geq 0,\end{aligned}$$

the last inequality is surely holds for $d \geq 3$.

(ii) We have to prove that

$$\left\lfloor \frac{\binom{d+n}{n} - (nd+1)}{d+1} \right\rfloor \geq r' + 2q'.$$

Since r' and q' are integers, the inequality above is equivalent to the following

$$\frac{\binom{d+n}{n} - (nd+1)}{d+1} \geq r' + 2q',$$

hence, it is enough to prove that

$$\binom{d+n}{n} - (nd+1) - (d+1)r' - 2(d+1)q' \geq 0.$$

We have

$$
\begin{aligned}
&\binom{d+n}{n} - (nd+1) - (d+1)r' - 2(d+1)q' \\
&= \binom{d+n}{n} - 2(d+1)\binom{d-1+n}{n} + 2(d+1)(n(d-1)+1) - (nd+1) \\
&\quad + 2(d+1)q + (d+1)(2d-1)r' \\
&\geq \binom{d+n}{n} - 2(d+1)\binom{d-1+n}{n} + 2(d+1)(n(d-1)+1) - (nd+1) \\
&\quad + 2(d+1)q + (d+1)(2d-1)\left\{\frac{\binom{d-1+n}{n} - (n(d-1)+1) - q}{d} - 1\right\} \\
&= \frac{1}{d}\left\{d\binom{d+n}{n} - (d+1)\binom{d-1+n}{n} - (n-1) + q(d+1) - d(d+1)(2d-1)\right\} \\
&= \frac{1}{d}\left\{(n-1)\binom{d-1+n}{n} - (n-1) + q(d+1) - d(d+1)(2d-1)\right\} \\
&\geq \frac{1}{d}\left\{(n-1)\binom{d-1+n}{n} - \binom{d-1+n}{n} - d(d+1)(2d-1)\right\} \\
&= \frac{1}{d}\left\{(n-2)\binom{d-1+n}{n} - d(d+1)(2d-1)\right\}.
\end{aligned}
$$

For $n \geq 5$, we get

$$
\begin{aligned}
&(n-2)\binom{d-1+n}{n} - d(d+1)(2d-1) \\
&\quad \geq 3\binom{d+4}{5} - d(d+1)(2d-1) \\
&\quad = \frac{1}{40}d(d+1)((d+2)(d+3)(d+4) - 80d + 40) \geq 0,
\end{aligned}
$$

it is quite immediate to check that the last inequality holds for all $d \geq 3$, hence we are done in the case $n \geq 5$.

For $n = 4$, we get

$$
\begin{aligned}
&(n-2)\binom{d-1+n}{n} - d(d+1)(2d-1) \\
&\quad = \frac{1}{12}d(d+1)((d+2)(d+3) - 24d + 12) \\
&\quad = \frac{1}{12}d(d+1)(d-1)(d-18),
\end{aligned}
$$

which is positive for all $d \geq 18$. This is what we wanted to show for $n = 4$ and $d \geq 18$, then we are left with $5 \leq d \leq 17$. Now by a direct computation we get the desired inequality $r - r' - 2q' \geq 0$ in the case $n = 4$ with $5 \leq d \leq 17$ as follows:

d	5	6	7	8	9	10	11	12	13	14	15	16	17
$r-r'-2q'$	7	9	2	10	9	10	17	9	30	34	17	35	32

□

Lemma 8.2 *Let $n \geq 5$ and $d \geq 3$. With the notations as in Lemma 8.1, let*

$$\bar{r} = \left\lfloor \frac{\binom{d+n-2}{n-2} - (n-1) - r + r'}{d} \right\rfloor,$$
$$\bar{q} = \binom{d+n-2}{n-2} - (n-1) - \bar{r}d - r + r'.$$

Then

(i) $\bar{r} \geq 0$;
(ii) $\bar{r} \leq r - r' - 2q'$;
(iii) $r' \geq q' + \bar{q}$;
(iv) $\binom{d+n-2}{n-1} - (r - r' - \bar{r})d = r' - q' - \bar{q} + 1.$

Proof (i) We will verify that

$$\binom{d+n-2}{n-2} - (n-1) - r + r' \geq 0.$$

Since

$$r \leq \frac{\binom{d+n}{n} - (nd+1)}{d+1}; \quad r' \geq \frac{\binom{d-1+n}{n} - (n(d-1)+1) - q}{d} - 1,$$

we have

$$\begin{aligned} r - r' \leq & \frac{1}{d+1}\left\{\binom{d+n}{n} - (nd+1)\right\} \\ & -\frac{1}{d}\left\{\binom{d-1+n}{n} - (n(d-1)+1) - q\right\} + 1 \\ = & \frac{1}{d(d+1)}\left\{d\binom{d+n}{n} - (d+1)\binom{d-1+n}{n} - (n-1) + q(d+1)\right\} + 1 \\ = & \frac{1}{d(d+1)}\left\{(n-1)\binom{d-1+n}{n} - (n-1) + q(d+1) + d(d+1)\right\}. \end{aligned}$$

Then we get

$$\begin{aligned}&\binom{d+n-2}{n-2}-(n-1)-r+r'\\&\geq\binom{d+n-2}{n-2}-(n-1)\\&\quad-\frac{1}{d(d+1)}\left\{(n-1)\binom{d-1+n}{n}-(n-1)+2d(d+1)\right\}\\&=\frac{\mathbf{A}}{d(d+1)},\end{aligned}$$

where

$$\mathbf{A}=d(d+1)\binom{d+n-2}{n-2}-(n-1)\binom{d-1+n}{n}-(n-1)(d^2+d-1)-2d(d+1).$$

A straightforward computation, yields

$$\mathbf{A}=\frac{1}{n}\binom{d+n-2}{n-2}(nd^2-d^2+d)-(n-1)(d^2+d-1)-2d(d+1).$$

Since $n\geq 5$, $d\geq 3$ we have $n(n-1)\leq\binom{d+n-2}{n-2}$, and from here it follows

$$\begin{aligned}\mathbf{A}&\geq(n-1)(nd^2-d^2+d-(d^2+d-1))-2d(d+1)\\&=d^2(n^2-3n)-2d+n-1\geq 0,\end{aligned}$$

which completes the proof.

(ii) In order to prove that $\bar{r}\leq r-r'-2q'$, it suffices to prove that

$$\frac{\binom{d+n-2}{n-2}-(n-1)-r+r'}{d}\leq r-r'-2q'+1,$$

which is equivalent to the following

$$r(d+1)-r'(d+1)-\binom{d+n-2}{n-2}+(n-1)-2q'd+d\geq 0.$$

From the definitions of r and r', moreover the inequality $q'\leq d-1$, we get

$$\begin{aligned}&r(d+1)-r'(d+1)-\binom{d+n-2}{n-2}+(n-1)-2q'd+d\\&\geq\binom{d+n}{n}-(nd+1)-(d+1)-\frac{d+1}{d}\left\{\binom{d-1+n}{n}-(n(d-1)+1)-q\right\}\\&\quad-\binom{d+n-2}{n-2}+(n-1)-2d^2+3d,\end{aligned}$$

which, by an easy computation, is equal to

$$\frac{1}{d}\left\{(n-1)\binom{n+d-2}{n}+n(d-1)-2d^3+2d^2-2d+1+q(d+1)\right\}.$$

Now we observe that

$$\begin{aligned}&(n-1)\binom{n+d-2}{n}+n(d-1)-2d^3+2d^2-2d+1+q(d+1)\\&\geq (n-1)\binom{n+d-2}{n}+n(d-1)-2d^3+2d^2-2d+1,\end{aligned}$$

hence, we will be done if we prove that

$$(n-1)\binom{n+d-2}{n}+n(d-1)-2d^3+2d^2-2d+1\geq 0. \tag{8}$$

For $n\geq 6$, we have

$$\begin{aligned}&(n-1)\binom{n+d-2}{n}+n(d-1)-2d^3+2d^2-2d+1\\&\geq 5\binom{d+4}{6}-(2d^3-2d^2-4d+5),\end{aligned}$$

which, for $d\geq 3$, is positive, as we wanted.

For $n=5$, the inequality (8) becomes:

$$4\binom{d+3}{5}-(2d^3-2d^2-3d+4)\geq 0,$$

which is true for $d\geq 5$, so we are left with $d=3,4$ in the case of $n=5$. But direct computations show that also these cases satisfy the required inequality $\bar{r}\leq r-r'-2q'$. More precisely, if $d=3$, we have $\bar{r}=3$ and $r-r'-2q'=5$; if $d=4$, we have $\bar{r}=5$ and $r-r'-2q'=11$.

(iii) We want to prove that $q'+\bar{q}\leq r'$. By the inequalities $q',\bar{q}\leq d-1$, which implies $q'+\bar{q}\leq 2d-2$, and also by the following one

$$r'\geq \frac{\binom{d-1+n}{n}-(n(d-1)+1)-q}{d}-1,$$

it is enough to prove that

$$\frac{\binom{d-1+n}{n}-(n(d-1)+1)-q}{d}-1\geq 2d-2,$$

i.e.

$$\binom{d-1+n}{n} - (n(d-1)+1) - q - 2d^2 + d \geq 0.$$

Using $q \leq d$, we have to show that

$$\binom{d-1+n}{n} - (n(d-1)+1) - 2d^2 \geq 0,$$

or, equivalently,

$$\binom{d-1+n}{n} - n(d-1) \geq 2d^2 + 1. \tag{9}$$

Notice that the function $\binom{d-1+n}{n} - n(d-1)$ is an increasing function in n.

For $n = 5$, the inequality (9) becomes

$$\binom{d+4}{5} \geq 2d^2 + 5d - 4,$$

which holds for $d \geq 4$. So it remains to check $q' + \bar{q} \leq r'$ in the case of $d = 3$ with $n = 5$. In this case we can directly compute that $r' = 3$, $q' = 1$, $\bar{q} = 0$. Hence the case $n = 5$ is done.

For $n = 6$, the inequality (9) becomes

$$\binom{d+5}{6} \geq 2d^2 + 6d - 5,$$

which holds for $d \geq 4$. So we are left with $d = 3$. A direct computation in the case of $d = 3$ with $n = 6$ yields that $r' = 4$, $q' = 2$, $\bar{q} = 0$. So we are done for $n = 6$.

Finally, for $n = 7$, the inequality (9) becomes

$$\binom{d+6}{7} \geq 2d^2 + 7d - 6,$$

which is true for any $d \geq 3$. Now, since $\binom{d-1+n}{n} - n(d-1)$ is an increasing function in n, we have proved (9) for all $n \geq 7$ and $d \geq 3$. That finishes the proof of part (iii).

(iv) We must check that

$$\binom{d+n-2}{n-1} - (r - r' - \bar{r})d = r' - q' - \bar{q} + 1,$$

that is

$$\binom{d+n-2}{n-1} + (r'd + q') + (\bar{r}d + \bar{q} - r') = rd + 1. \tag{10}$$

From the definitions of $q', \bar{q}$ we have

$$r'd + q' = \binom{d-1+n}{n} - n(d-1) - 1 - q;$$
$$\bar{r}d + \bar{q} - r' = \binom{d+n-2}{n-2} - (n-1) - r.$$

Now using these equalities and an easy computation yields

$$\binom{d+n-2}{n-1} + (r'd + q') + (\bar{r}d + \bar{q} - r') = \binom{d+n}{n} - nd - r - q,$$

which, by recalling that $q = \binom{d+n}{n} - (nd+1) - r(d+1)$, is equal to $(rd+1)$, that is what we wanted (10). □

Lemma 8.3 *Let $d \geq 5$. Let r, r', q, q' be as in Lemma 8.1 in the case $n = 4$. Let*

$$\hat{r} = \left\lfloor \frac{(d+1)^2 - (d+2)q' - 2(r - r' - 2q')}{d-1} \right\rfloor,$$
$$\hat{q} = (d+1)^2 - (d+2)q' - (d-1)\hat{r} - 2(r - r' - 2q').$$

Then

(i) $\hat{r} \geq 0$;
(ii) $\hat{r} \leq r - r' - 2q'$;
(iii) $\hat{q} \leq r'$;
(iv) $q' + \hat{r} \leq d$;
(v) $r' - \hat{q} = \binom{d+1}{3} - 4 - (d-1)(r - r' - \hat{r} - q')$.

Proof (i) We need to show that

$$(d+1)^2 - (d+2)q' - 2(r - r' - 2q') \geq 0,$$

that is

$$(d+1)^2 - (d-2)q' - 2(r - r') \geq 0.$$

Recall:

$$r = \left\lfloor \frac{\binom{d+4}{4} - 4d - 1}{d+1} \right\rfloor, \quad q = \binom{d+4}{4} - 4d - 1 - r(d+1);$$
$$r' = \left\lfloor \frac{\binom{d+3}{4} - 4d + 3 - q}{d} \right\rfloor, \quad q' = \binom{d+3}{4} - 4d + 3 - q - r'd.$$

Let us start by computing $(d-2)q' + 2(r-r')$:

$$\begin{aligned}(d-2)q' + 2r - 2r' &= (d-2)\binom{d+3}{4} - (d-2)(4d-3)\\ &\quad -(d-2)q - (d^2-2d+2)r' + 2r\\ &\leq \frac{\mathbf{A}}{d(d+1)},\end{aligned}$$

where

$$\begin{aligned}\mathbf{A} &= (d^2+d)(d-2)\binom{d+3}{4} - (d^2+d)(d-2)(4d-3)\\ &\quad - (d^2+d)(d-2)q - (d+1)(d^2-2d+2)\left\{\binom{d+3}{4} - 4d + 3 - q\right\}\\ &\quad + 2d\left\{\binom{d+4}{4} - 4d - 1 - (d+1)\right\}\\ &= 2d\binom{d+4}{4} - 2(d+1)\binom{d+3}{4} - 2(d^2+d+3) + 2q(d+1)\\ &= 6\binom{d+3}{4} - 2(d^2+d+3) + 2q(d+1).\end{aligned}$$

Therefore we get

$$\begin{aligned}(d-2)q' + 2(r-r') &\leq \frac{\mathbf{A}}{d(d+1)}\\ &= \frac{(d+2)(d+3)}{4} - \frac{2(d^2+d+3)}{d^2+d} + \frac{2q}{d},\end{aligned}$$

then, by noting that $\frac{2(d^2+d+3)}{d^2+d} \geq 2$ and that $q \leq d$, it immediately follows

$$(d-2)q' + 2(r-r') \leq \frac{(d+2)(d+3)}{4}.$$

Now from here we have

$$\begin{aligned}(d+1)^2 - (d-2)q' - 2(r-r') &\geq (d+1)^2 - \frac{(d+2)(d+3)}{4}\\ &= \frac{3d^2+3d-2}{4} \geq 0,\end{aligned}$$

and this finishes the proof.

(ii) In order to check that $\hat{r} \leq r - r' - 2q'$, it suffices to check that

$$\frac{(d+1)^2 - (d+2)q' - 2(r-r'-2q')}{d-1} \leq r - r' - 2q' + 1,$$

that is

$$(d+1)^2 - (d+2)q' - 2(r - r' - 2q') \leq (d-1)(r - r' - 2q') + (d-1),$$

or, equivalently

$$(d+1)(r - r') - dq' - (d+1)^2 + (d-1) \geq 0.$$

Again, using the definitions of r and r', moreover the inequality $q' \leq d-1$, one gets

$$\begin{aligned}
&(d+1)(r-r') - dq' - (d+1)^2 + (d-1) \\
&\quad \geq (d+1)r - (d+1)r' - 2(d^2+1) \\
&\quad \geq \binom{d+4}{4} - (4d+1) - (d+1) \\
&\qquad - \frac{d+1}{d}\left\{\binom{d+3}{4} - 4d + 3 - q\right\} - 2(d^2+1),
\end{aligned}$$

which, by a short computation, is equal to

$$\frac{1}{d}\left\{3\binom{d+3}{4} - (2d^3 + d^2 + 3d + 3) + q(d+1)\right\}.$$

Now we have

$$\begin{aligned}
&3\binom{d+3}{4} - (2d^3 + d^2 + 3d + 3) + q(d+1) \\
&\quad \geq 3\binom{d+3}{4} - (2d^3 + d^2 + 3d + 3) \\
&\quad = \frac{1}{8}(d^4 - 10d^3 + 3d^2 - 18d - 24),
\end{aligned}$$

which, in fact for $d \geq 10$ is positive, as required. Then it remains to check that the cases $5 \leq d \leq 9$ satisfy $\hat{r} \leq r - r' - 2q'$. Computing each of these cases, we get the conclusion:

d	$\hat{r}$	$r - r' - 2q'$
5	5	7
6	6	9
7	2	2
8	5	10
9	4	9

(iii) To prove $\hat{q} \leq r'$, by noting that $\hat{q} \leq d-2$, it is enough to prove

$$\frac{\binom{d+3}{4} - 4d + 3 - q}{d} \geq d - 1,$$

i.e.

$$\binom{d+3}{4} - 4d + 3 - q - d(d-1) \geq 0. \tag{11}$$

Observe that

$$\begin{aligned} &\binom{d+3}{4} - 4d + 3 - q - d(d-1) \\ &\geq \binom{d+3}{4} - 4d + 3 - d - d(d-1) \\ &= \binom{d+3}{4} - (d^2 + 4d - 3). \end{aligned}$$

For $d \geq 5$, it is immediate to see that

$$\binom{d+3}{4} - (d^2 + 4d - 3) \geq 0,$$

which gives (11).

(iv) We will show that $d - q' - \hat{r} \geq 0$. We have

$$\begin{aligned} d - q' - \hat{r} &\geq d - q' - \frac{1}{d-1}((d+1)^2 - (d-2)q' - 2r + 2r') \\ &= \frac{1}{d-1}(2r - 2r' - q' - 3d - 1), \end{aligned}$$

moreover,

$$\begin{aligned} &2r - 2r' - q' - 3d - 1 \\ &= 2r + (d-2)r' - \binom{d+3}{4} + d - 4 + q \\ &\geq \frac{\mathbf{A}}{d(d+1)}, \end{aligned}$$

where,

$$\begin{aligned} \mathbf{A} = 2d\binom{d+4}{4} &- 2d(4d+1) - 2d(d+1) + (d-2)(d+1)\binom{d+3}{4} \\ &- (d-2)(d+1)(4d-3) - (d-2)(d+1)q - d(d+1)(d-2) \\ &- (d^2+d)\binom{d+3}{4} + (d^2+d)(d-4) + (d^2+d)q. \end{aligned}$$

One can easily find that

$$\begin{aligned}\mathbf{A} &= 2d\binom{d+4}{4} - 2(d+1)\binom{d+3}{4} \\ &\quad - (4d^3+5d^2+d+6) + 2(d+1)q \\ &= 6\binom{d+3}{4} - (4d^3+5d^2+d+6) + 2(d+1)q \\ &\geq 6\binom{d+3}{4} - (4d^3+5d^2+d+6),\end{aligned}$$

which is positive for $d \geq 11$, hence we are left with $5 \leq d \leq 10$. Now by direct calculations we get $q' + \hat{r} \leq d$ in these cases as follows:

d	q'	$\hat{r}$	$q'+\hat{r}$
5	0	5	5
6	0	6	6
7	5	2	7
8	2	5	7
9	4	4	8
10	5	4	9

(v) We have to verify that

$$r' - \hat{q} = \binom{d+1}{3} - 4 - (d-1)(r - r' - \hat{r} - q'),$$

that is

$$(d-1)\hat{r} + \hat{q} + (d-2)r' + (d-1)q' = (d-1)r - \binom{d+1}{3} + 4. \tag{12}$$

Rewrite the left hand side as

$$((d-1)\hat{r} + \hat{q} + (d-2)q' - 2r') + (dr' + q').$$

Recalling that, from the definitions of q' and $\hat{q}$,

$$\begin{aligned}(d-1)\hat{r} + \hat{q} + (d-2)q' - 2r' &= (d+1)^2 - 2r; \\ dr' + q' &= \binom{d+3}{4} - 4d + 3 - q,\end{aligned}$$

the left hand side of (12) becomes:

$$\begin{aligned}
&(d+1)^2 - 2r + \binom{d+3}{4} - 4d + 3 - q \\
&= (d+1)^2 - 2r + \binom{d+3}{4} - 4d + 3 - \binom{d+4}{4} + 4d + 1 + (d+1)r \\
&= (d+1)^2 - \binom{d+3}{3} + 4 + (d-1)r \\
&= -\binom{d+1}{3} + 4 + (d-1)r,
\end{aligned}$$

and we are done. □

References

1. Aladpoosh, T., Ballico, E.: Postulation of disjoint unions of lines and a multiple point. Rend. Semin. Mat. Univ. Politec. Torino **72**(3–4), 127–145 (2014)
2. Alexander, J., Hirschowitz, A.: Polynomial interpolation in several variables. J. Algebr. Geom. **4**(2), 201–222 (1995)
3. Ballico, E.: On the Hilbert functions of disjoint unions of a linear space and many lines in $\mathbb{P}^n$. Int. Math. Forum **5**(16), 787–798 (2010)
4. Ballico, E.: Postulation of disjoint unions of lines and a few planes. J. Pure Appl. Algebra **215**(4), 597–608 (2011)
5. Ballico, E.: Postulation of disjoint unions of lines and a multiple point II. Mediterr. J. Math. **13**(4), 1449–1463 (2016)
6. Ballico, E.: On the maximal rank of a general union of a multiple linear space and a generic rational curve. Bol. Soc. Mat. Mex. **22**(1), 13–31 (2016)
7. Carlini, E., Catalisano, M.V., Geramita, A.V.: Bipolynomial Hilbert functions. J. Algebra **324**(4), 758–781 (2010)
8. Carlini, E., Catalisano, M.V., Geramita, A.V.: 3-Dimensional sundials. Cent. Eur. J. Math. **9**(5), 949–971 (2011)
9. Carlini, E., Catalisano, M.V., Geramita, A.V.: Subspace arrangements, configurations of linear spaces and the quadrics containing them. J. Algebra **362**(3), 70–83 (2012)
10. Carlini, E., Catalisano, M.V., Geramita, A.V.: On the Hilbert function of lines union one non-reduced point. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **XV**, 69–84 (2016)
11. Ciliberto, C.: Geometric aspects of polynomial interpolation in more variables and of Waring's problem. In: Proceedings of the European Congress of Mathematics, Barcelona (2000); in: Progress in Mathematics, pp. 289–316. Brikhaüser (2001)
12. Cooper, S., Harbourne, B.: Regina lectures on fat points. In: Cooper, S., Sather-Wagstaff, S. (eds.) Connections Between Algebra, Combinatorics and Geometry. Springer Proceedings in Mathematics and Statistics, vol. 76, pp. 147–187. Springer, New York (2014)
13. Derksen, H.: Hilbert series of subspace arrangements. J. Pure Appl. Algebra **209**(1), 91–98 (2007)
14. Derksen, H., Sidman, J.: A sharp bound for the Castelnuovo–Mumford regularity of subspace arrangements. Adv. Math. **172**(2), 151–157 (2002)
15. Dumnicki, M., Harbourne, B., Szemberg, T., Tutaj-Gasińska, H.: Linear subspaces, symbolic powers and Nagata type conjectures. Adv. Math. **252**, 471–491 (2014)
16. Fatabbi, G., Harbourne, B., Lorenzini, A.: Inductively computable unions of fat linear subspaces. J. Pure Appl. Algebra **219**, 5413–5425 (2015)
17. Fulton, W.: Intersection Theory. Springer, Berlin (1984)
18. Geramita, A.V., Maroscia, P., Roberts, L.G.: The Hilbert function of a reduced k-algebra. J. Lond. Math. Soc. (2) **28**(3), 443–452 (1983)

19. Geramita, A.V., Orecchia, F.: On the Cohen–Macaulay type of s lines in $\mathbb{A}^{n+1}$. J. Algebra **70**, 116–140 (1981)
20. Harbourne, B., Roé, J.: Linear systems with multiple base points in $\mathbb{P}^2$. Adv. Geom. **4**(1), 41–59 (2004)
21. Hartshorne, R.: Algebraic Geometry. Springer, New York (1977)
22. Hartshorne, R., Hirschowitz, A.: Droites en position générale dans l'espace projectif. In: Aroca, J.M., Buchweitz, R., Giusti, M., Merle, M. (eds.) Algebraic Geometry. Lecture Notes in Mathematics, vol. 961, pp. 169–188. Springer, Berlin (1982)
23. Hirschowitz, A.: Sur la postulation générique des courbes rationnelles. Acta Math. **146**, 209–230 (1981)