On the lower bound of the sum of the algebraic connectivity of a graph and its complement

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Abstract

For a graph G, let $\mu_2(G)$ denote its second smallest Laplacian eigenvalue. It was conjectured that $\mu_2(G)+\mu_2(\overline{G}) \geq 1$, where $\overline{G}$ is the complement of G. This conjecture has been proved for various families of graphs. Here, we prove this conjecture in the general case. Also, we will show that $\max\{\mu_2(G), \mu_2(\overline{G})\} \geq 1 - O(n^{-\frac{1}{3}})$, where n is the number of vertices of G.

AMS Classification: 05C50
Keywords: Laplacian eigenvalues of graphs, Nordhaus-Gaddum type inequalities, Effective resistance, Laplacian spread

1 Introduction

Let G be a simple graph with n vertices. The Laplacian of G is defined to be $L := D - A$, where A is the the adjacency matrix of G and D is the diagonal matrix of vertex degrees. The eigenvalues of L,

$$0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G),$$

expose various properties of G. The "algebraic connectivity" of a graph $\mu(G) := \mu_2(G)$ is an efficient measure for connectivity of graphs.

The Laplacian spread of a graph G is defined to be $\mu_n(G) - \mu_2(G)$. It was conjectured [YL12, ZSH11] that this quantity is at most $n - 1$, or equivalently, $\mu_2(G) + \mu_2(\overline{G})$ is at least 1 (since $\mu_n(G) = n - \mu_2(\overline{G})$).

Conjecture. For any graph G of order $n \geq 2$, the following holds:

$$\mu(G) + \mu(\overline{G}) \geq 1,$$

with equality if and only if G or $\overline{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n - 1$.

This conjecture was proved for trees [FXWL08], unicyclic graphs [BTF09], bicyclic graphs [FLT10], tricyclic graphs [CW09], cactus graphs [Liu10], quasi-tree graphs [XM11], graphs with diameter not equal to 3 [ZSH11], bipartite graphs [ATR14], and K_3-free graphs [CD16]. Also, [AAMM18] provided a constant lower bound for $\mu(G) + \mu(\overline{G})$, by proving that $\max\{\mu(G), \mu(\overline{G})\} \geq \frac{2}{5}$.

Here, we prove the conjecture, in the general case. The main idea of the proof is as follows. Suppose that x and y, respectively, are eigenvectors of $L(G)$ and $L(\overline{G})$, corresponding to the eigenvalues $\mu_2(G)$ and $\mu_2(\overline{G})$, with $\|x\|_2 = \|y\|_2 = 1$. We have $\sum_i x_i = \sum_i y_i = 0$ and

$$\begin{aligned}
\mu(G) + \mu(\overline{G}) &= x^T L(G)x + y^T L(\overline{G})y \\
&= \sum_{\{i,j\}\in E(G)} (x_i - x_j)^2 + \sum_{\{i,j\}\notin E(G)} (y_i - y_j)^2 \\
&\geq \sum_{i<j} \min\{(x_i - x_j)^2, (y_i - y_j)^2\}.
\end{aligned}$$

So, it is enough to show that

$$\sum_{i<j} \min\{(x_i - x_j)^2, (y_i - y_j)^2\} \geq 1.$$

Now, suppose that M is the maximum of all of numbers $|x_i - x_j|^2$ and $|y_i - y_j|^2$, for every $1 \leq i, j \leq n$. A simple algebraic identity (see Lemma 4), shows that $\sum_{i<j} \min\{|x_i - x_j|^2, |y_i - y_j|^2\} \geq \frac{1}{M}$. So, if $M \leq 1$, the proof is complete. It remains to consider the case $M > 1$. We manage this case with some basic properties of the effective resistances between pairs of vertices of G and a useful characterization of the algebraic connectivity of graphs due to Fiedler (see Lemma 2).

Moreover, here, we give an asymptotic lower bound for the maximum of $\mu(G)$ and $\mu(\overline{G})$ by proving the following theorem.

Theorem. For all simple graphs G with n vertices,

$$\max\{\mu(G), \mu(\overline{G})\} \geq 1 - O(n^{-\frac{1}{3}}).$$

Also, for each $n \geq 4$, there is a graph G which has n vertices such that the maximum of $\mu(G)$ and $\mu(\overline{G})$ is less than 1. So, if c_n denote the minimum of $\max\{\mu(G), \mu(\overline{G})\}$ over all graphs G with n vertices, then $c_n = 1 - o(1)$.

The organization of the remaining of the paper is as follows. In Section 2, we gives some preliminaries and notations, which is necessary. In Section 3, we present two main results of the paper.

2 Preliminaries and notations

Throughout this paper G is a simple graph, with $n \geq 2$ vertices $V(G) := \{v_1, \ldots, v_n\}$ and edges $E(G)$. The notation $\{i, j\} \in E(G)$, for $i, j \in \{1, \ldots, n\}$, means that v_i and v_j are adjacent in G. Also, $A(G)$ denotes the adjacency matrix of G. $D(G)$ is the diagonal matrix of vertex degree. We denote the Laplacian matrix of G by $L(G) := D(G) - A(G)$ and its eigenvalues by $0 = \mu_1(G) \le \mu_2(G) \le \cdots \le \mu_n(G) \le n$ and the algebraic connectivity of G by $\mu(G) := \mu_2(G)$. $\overline{G}$ and $\operatorname{diam}(G)$, respectively, denote the complement of G and the diameter of G. If v is a vertex of G, then $N_G(v)$ denotes the set of vertices which are adjacent to v in G. Recall that for each $1 \le i < n$, we have $\mu_{i+1}(G) = n - \mu_{n+1-i}(\overline{G})$.

The "effective resistance" between two vertices v_r and v_s in a graph G is denoted by $R^G_{r,s}$ and defined by:

$$\frac{1}{R^G_{r,s}} := \min \sum_{\{i,j\}\in E(G)} (x_i - x_j)^2,$$

where the minimum runs over all $x \in \mathbb{R}^n$, with $x_r - x_s = 1$. So for an arbitrary vector $x \in \mathbb{R}^n$ and $1 \le r < s \le n$, we have

$$\sum_{\{i,j\}\in E(G)} (x_i - x_j)^2 \ge \frac{(x_r - x_s)^2}{R^G_{r,s}}.$$

(In the nontrivial case $x_r - x_s \neq 0$, by dividing the vector x by $x_r - x_s$, we can assume $x_r - x_s = 1$ and use the defining formula for $R^G_{r,s}$.)

In this paper we only use some very basic properties of the effective resistance which are listed in the following lemma:[1]

Lemma 1. *Let $G = (V, E)$ be a simple graph with $n \ge 2$ vertices. Then the following holds:*

(a) For every subgraph H of G which contains the vertices v_1 and v_2, $R^G_{1,2} \le R^H_{1,2}$.

(b) "Total resistance of parallel circuits": Let G_1 and G_2 be two subgraphs of G, such that $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and $E(G)$ be the disjoint union of $E(G_1)$ and $E(G_2)$. Then

$$\frac{1}{R^G_{1,2}} = \frac{1}{R^{G_1}_{1,2}} + \frac{1}{R^{G_2}_{1,2}}.$$

(c) "Total resistance of series circuits": For $n \ge 3$, let G_1 and G_2 be two induced subgraphs of G, such that $v_1 \in V(G_1)$, $v_3 \in V(G_2)$ and $V(G_1) \cap V(G_2) = \{v_2\}$. Suppose also that $E(G)$ be the disjoint union of $E(G_1)$ and $E(G_2)$. Then

$$R^G_{1,3} = R^{G_1}_{1,2} + R^{G_2}_{2,3}$$

[1] For more information about the effective resistance and its relation to the algebraic connectivity of graphs, see [ESVM+11]. For the equivalence of the above definition of effective resistance and the standard definition by the concepts of electrical circuits and Kirchhoff's circuit laws, see [Lya99].

Proof. The parts (a) and (b) are immediate from the defining equation of the effective resistance. For (c), we have:

$$\begin{aligned}
\frac{1}{R^G_{1,3}} &= \min_{x\in\mathbb{R}^n, x_1=0, x_3=1} \sum_{\{i,j\}\in E(G)} (x_i - x_j)^2 \\
&= \min_{x\in\mathbb{R}^n, x_1=0, x_3=1} \sum_{\{i,j\}\in E(G_1)} (x_i - x_j)^2 + \sum_{\{i,j\}\in E(G_2)} (x_i - x_j)^2 \\
&= \min_{x_2\in\mathbb{R}, x_1=0, x_3=1} \frac{(x_1 - x_2)^2}{R^{G_1}_{1,2}} + \frac{(x_2 - x_3)^2}{R^{G_2}_{2,3}} \\
&= \min_{x_2\in\mathbb{R}} \frac{x_2^2}{R^{G_1}_{1,2}} + \frac{(x_2 - 1)^2}{R^{G_2}_{2,3}} \\
&= \frac{1}{R^{G_1}_{1,2} + R^{G_2}_{2,3}}.
\end{aligned}$$

(The equality of the second and the third line, uses the fact that G_1 and G_2 have only the vertex v_2 in common.) ☐

We conclude this section with a useful lemma, due to Fiedler.

Lemma 2 ([Fie75])**.** *Let $G = (V, E)$ be a connected graph. Then $\mu(G)$ is positive and equal to the minimum of the function*

$$\varphi(x) := n\frac{\sum_{\{i,j\}\in E(G)}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2}$$

over all nonconstant n-tuples $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (i.e. n-tuples which are not of the form $x_i = c, i = 1, \ldots, n$).

Therefore $\mu(G)$ is the largest real number for which the following inequality holds for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$n \sum_{\{i,j\}\in E(G)} (x_i - x_j)^2 \geq \mu(G) \sum_{i<j} (x_i - x_j)^2. \quad (1)$$

3 Main results

3.1 Sum of the algebraic connectivity of a graph and its complement

Theorem 1. *For any graph G of order $n \geq 2$, the following holds:*

$$\mu(G) + \mu(\overline{G}) \geq 1,$$

with equality if and only if G or $\overline{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n - 1$.

Note that for any connected graph G, $\mu(G)$ is a positive number. So, when G is a disconnected graph, Theorem 1 can be reformulated as the following lemma:

Lemma 3. *If G is a disconnected graph, then $\mu(\overline{G}) \geq 1$, with equality if and only if $\overline{G}$ is the join of an isolated vertex and a disconnected graph.*

Proof. We know that the biggest Laplacian eigenvalue of every graph is at most the number of its vertices and the Laplacian eigenvalues of G are the union of the eigenvalues of its connected components, where each of which has less than n vertices. So, $\mu_n(G) \leq n-1$ and $\mu(\overline{G}) = n - \mu_n(G) \geq 1$, with equality if and only if G is the disjoint union of an isolated vertex and a connected graph H with $n-1$ vertices, where $\mu_{n-1}(H) = n-1$. But, note that $\mu_{n-1}(H) = n-1-\mu_2(\overline{H})$. So, we have $\mu_{n-1}(H) = n-1$, if and only if $\overline{H}$ is disconnected. Therefore, we have equality $\mu(\overline{G}) = 1$, if and only if $\overline{G}$ is the join of an isolated vertex and a disconnected graph $\overline{H}$. □

Lemma 4. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$ with $\sum_i y_i = \sum_i x_i = 0$. Then*

$$\sum_{i<j}(x_i - x_j)^2(y_i - y_j)^2 \geq \|x\|^2\|y\|^2.$$

Proof. Denote $A = \sum_{i<j}(x_i - x_j)^2(y_i - y_j)^2$. We have

$$\begin{aligned}
2A &= \sum_{i\neq j}(x_i - x_j)^2(y_i - y_j)^2 \\
&= \sum_{i,j}(x_i - x_j)^2(y_i - y_j)^2 \quad \text{(The terms corresponding to values } i = j \text{ are zero)} \\
&= \sum_{i,j}(x_i^2 + x_j^2 - 2x_ix_j)(y_i^2 + y_j^2 - 2y_iy_j) \\
&= \sum_{i,j}(x_i^2y_i^2 + x_j^2y_j^2) + \sum_{i,j}(x_i^2y_j^2 + x_j^2y_i^2) + 4\sum_{i,j} x_ix_jy_iy_j \\
&\quad -2\sum_{i,j}\left(x_i^2y_iy_j + x_j^2y_iy_j + x_ix_jy_i^2 + x_ix_jy_j^2\right) \\
&= 2\sum_{i,j} x_i^2y_i^2 + 2\sum_{i,j} x_i^2y_j^2 + 4\sum_{i,j} x_ix_jy_iy_j - 4\sum_{i,j}\Big(x_i^2y_iy_j + x_ix_jy_i^2\Big) \\
&= 2n\sum_i x_i^2y_i^2 + 2\Big(\sum_i x_i^2\Big)\Big(\sum_j y_j^2\Big) + 4\Big(\sum_i x_iy_i\Big)\Big(\sum_j x_jy_j\Big) \\
&\quad -4\Big(\sum_i x_i^2y_i\Big)\Big(\sum_j y_j\Big) - 4\Big(\sum_i x_iy_i^2\Big)\Big(\sum_j x_j\Big) \\
&= 2n\sum_i x_i^2y_i^2 + 2\Big(\sum_i x_i^2\Big)\Big(\sum_i y_i^2\Big) + 4\Big(\sum_i x_iy_i\Big)^2 \quad \text{(by } \textstyle\sum_j x_j = \sum_j y_j = 0\text{)} \\
&\geq 2\|x\|^2\|y\|^2.
\end{aligned}$$

□

Lemma 5. *Suppose that G is a connected graph with $n \geq 2$ vertices $\{v_1, \dots, v_n\}$. Let $x = (x_1, \dots, x_n)$ be an eigenvector of $L(G)$ corresponding to $\mu_2(G)$. If*

$$x_1 = \max_i x_i, \quad x_2 = \min_i x_i,$$

and the distance between v_1 and v_2 is at most equal to 2, then $\mu(G) \geq 1$. In particular, for any graph with diameter less than 3, we have $\mu(G) \geq 1$.

Proof. Since x is orthogonal to the constant-entries vector $(1, \dots, 1)$, $\sum_i x_i = 0$. So, $x_2 < 0 < x_1$. Now, from $L(G)x = \mu x$, we get $\mu x_1 = \sum_{\{i,1\} \in E(G)} (x_1 - x_i)$. But, for each i, $x_1 - x_i \geq 0$. Thus, for every $v_i \sim v_1$, we have $x_1 - x_i \leq \mu x_1$, or equivalently, $x_i \geq (1-\mu)x_1$. Similarly, for every $v_j \sim v_2$, we have $x_j \leq (1-\mu)x_2$. Therefore, if $\mu < 1$, for every $v_i \sim v_1$ and $v_j \sim v_2$ we have $x_j < 0 < x_i$. So v_1 and v_2 are not adjacent and have no common neighbor. Thus, the distance between v_1 and v_2 is greater than 2. □

Proof of Theorem 1. According to Lemma 3, we can suppose that both G and $\overline{G}$ are connected. So, we have $n > 2$. Denote $\mu_2(G)$ and $\mu_2(\overline{G})$, respectively by μ and $\overline{\mu}$. Let $x = (x_1, \dots, x_n)$ be a normalized eigenvector of $L(G)$ corresponding to $\mu_2(G)$ and $y = (y_1, \dots, y_n)$ be a normalized eigenvector of $L(\overline{G})$ corresponding to $\mu_2(\overline{G})$. Note that y is also an eigenvector of $L(G)$ corresponding to $\mu_n(G)$. So x and y are two orthonormal vectors in $\mathbb{R}^n$ which are orthogonal to $e = (1, \dots, 1)$ (the eigenvector of $\mu_1(G) = 0$). Now, without loss of generality, we can suppose that $\max_{i<j} |x_i - x_j| \geq \max_{i<j} |y_i - y_j|$ and

$$x_1 = \max_{1 \leq i \leq n} x_i, \quad x_2 = \min_{1 \leq i \leq n} x_i.$$

Note that since $\sum_i x_i = 0$ and x is nonzero, we have $x_2 < 0 < x_1$.

Step 1 (the case $x_1 - x_2 < 1$). We have

$$\begin{aligned}
\mu + \overline{\mu} &= \sum_{\{i,j\} \in E(G)} (x_i - x_j)^2 + \sum_{\{i,j\} \in E(\overline{G})} (y_i - y_j)^2 \\
&\geq \sum_{i<j} \min\{(x_i - x_j)^2, (y_i - y_j)^2\} \\
&\geq \frac{\sum_{i<j} (x_i - x_j)^2 (y_i - y_j)^2}{\max_{i<j} \max\{(x_i - x_j)^2, (y_i - y_j)^2\}} \\
&\geq \frac{\|x\|^2 \|y\|^2}{(x_1 - x_2)^2} \qquad \text{(by Lemma 4)} \\
&= \frac{1}{(x_1 - x_2)^2}. \qquad \text{(since } x \text{ and } y \text{ are normalized)}
\end{aligned}$$

The first inequality is implied by the fact that, for every pair $i < j$, $\{i, j\}$ is an edge for exactly one of G or $\overline{G}$. For the second inequality, note that:

$$\begin{aligned}(x_i - x_j)^2(y_i - y_j)^2 = & \min\{(x_i - x_j)^2, (y_i - y_j)^2\} \max\{(x_i - x_j)^2, (y_i - y_j)^2\} \\ & \leq \min\{(x_i - x_j)^2, (y_i - y_j)^2\} \max_{k<l} \max\{(x_k - x_l)^2, (y_k - y_l)^2\}.\end{aligned}$$

Thus, in the case $x_1 - x_2 < 1$, we have $\mu + \overline{\mu} > 1$ and the proof is complete. So, we can suppose that $x_1 - x_2 \geq 1$.

Step 2 (the case $x_1 - x_2 \geq 1$). Since μ and $\overline{\mu}$ are positive, if one of them is greater than or equal to 1 then $\mu + \overline{\mu} > 1$. So, we can suppose that $\mu, \overline{\mu} < 1$. Let d be the distance between v_1 and v_2 in G.

Step 2.1 ($d \leq 2$ or $d > 3$). If $d \leq 2$ then, by Lemma 5, we have $\mu \geq 1$. On the other hand, if $d > 3$ then $\operatorname{diam}(G) > 3$. So the diameter of $\overline{G}$ is at most equal to 2. Thus, by Lemma 5, $\overline{\mu} \geq 1$. Therefore, we can suppose that $d = 3$.

Step 2.2 ($d = 3$). Now, suppose that $s \geq 1$ is the maximum number of vertex-disjoint paths with length 3 between two vertices v_1, v_2 in G.

Let S be the union of the vertices of s disjoint paths between v_1 and v_2 with length 3. Denote $S_1 := S \cap N_G(v_2)$ and $S_2 := S \cap N_G(v_1)$ (see Figure 1). Note that because v_1 and v_2 have no common neighbor in G, the vertices of S_1 are not adjacent to v_1 and the vertices of S_2 are not adjacent to v_2. Also, $|S_1| = |S_2| = s$ and $S = S_1 \cup S_2 \cup \{v_1, v_2\}$. Therefore, if we denote $A := N_G(v_1) \setminus S$, $B := N_G(v_2) \setminus S$, and $C := V(G) \setminus (A \cup B \cup S)$ then $\{A, B, C, S_1, S_2, \{v_1, v_2\}\}$ is a partition of the vertices of G and because by the maximality of s, there is no path of length at most 3 between v_1 and v_2 in $G \setminus (S_1 \cup S_2)$, no vertex of $A \cup \{v_1\}$ is adjacent to any of the vertices of $B \cup \{v_2\}$.

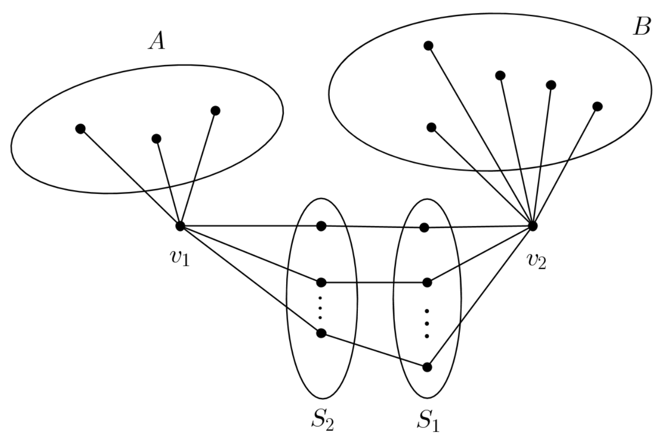

Figure 1: The subsets S_1, S_2, A and B of the vertices of G.

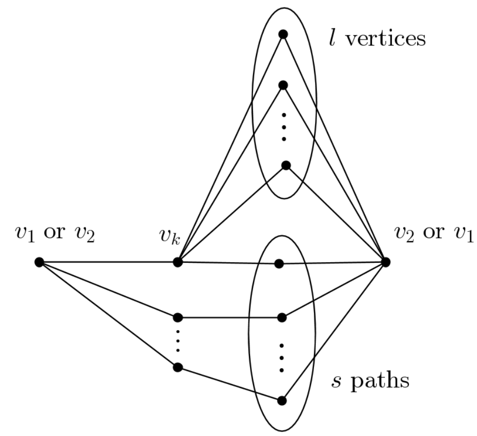

Figure 2: G_1: A subgraph of G.

Now, if we let $a := |A|, b := |B|$ and $c := |C|$, then

$$n = a + b + c + 2s + 2. \tag{2}$$

Furthermore we can suppose that $a \leq b$.

Suppose that l is the maximum of $|N_G(v) \cap A|$ for all $v \in S_1$ and $|N_G(u) \cap B|$ for all $u \in S_2$. So G contains a subgraph G_1, as illustrated in Figure 2. We have

$$\mu \geq \frac{(x_1 - x_2)^2}{R^G_{1,2}} \geq \frac{1}{R^G_{1,2}}.$$

By part (a) of Lemma 1, $R^G_{1,2} \leq R^{G_1}_{1,2}$. By multiple usage of the rules of total resistance for parallel and series circuits in Lemma 1, we have:

$$\frac{1}{R^{G_1}_{1,2}} = \frac{1}{1 + \frac{1}{\underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{l+1 \text{ times}}}} + \underbrace{\frac{1}{3} + \cdots + \frac{1}{3}}_{s-1 \text{ times}} + = \frac{1}{1 + \frac{2}{l+1}} + \frac{s-1}{3} = \frac{l+1}{l+3} + \frac{s-1}{3}.$$

(The terms $\frac{1}{2}$ correspond to $l+1$ parallel paths of length 2 between v_k and v_2 or v_1 in G_1. Also the terms $\frac{1}{3}$ correspond to $s-1$ parallel paths of length 3 not including v_k between v_1 and v_2 in G_1, as in the Figure 2).

Finally we have a lower bound for μ in terms of s and l:

$$\mu \geq \frac{s-1}{3} + \frac{l+1}{l+3}. \tag{3}$$

Next, we give a lower bound for $\overline{\mu}$. Note that $\overline{G}$ contains the edges of the graph illustrated in Figure 3, which hereafter is called G_2. Also, according to the definition of l, each vertex of S_1 are adjacent to at most l vertices of A in G and each vertex of S_2 are adjacent to at most l vertices of B in G.

Now, we define the subgraphs H_1, H_2 and H_3 of $\overline{G}$ as follows: (see figure 4)

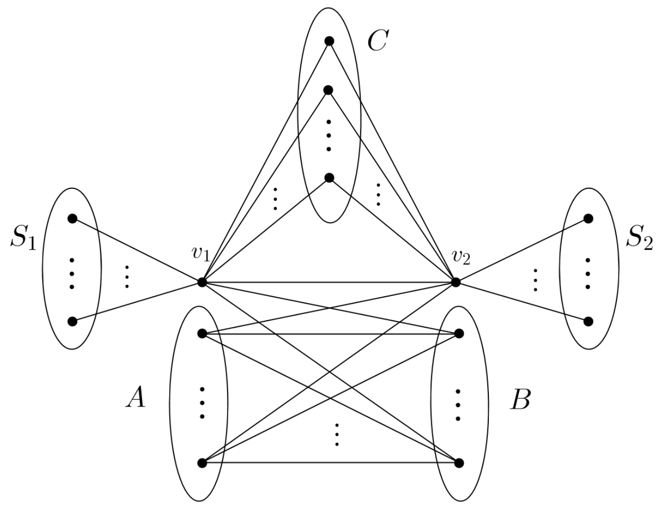

Figure 3: G_2: A subgraph of $\overline{G}$.

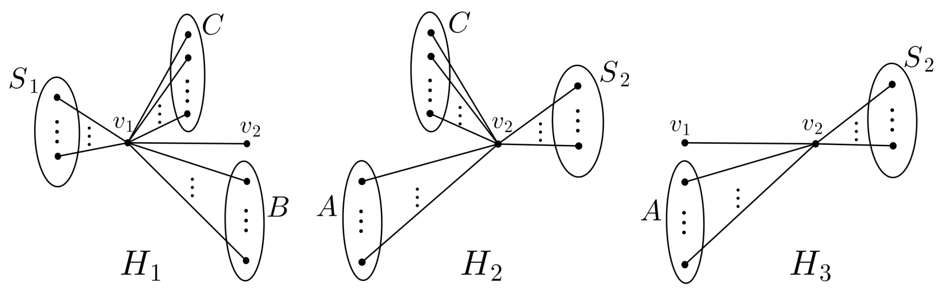

Figure 4: Graphs H_1, H_2 and H_3.

1. H_1 is the union of the edges which join v_1 to the vertices of $\{v_2\} \cup B \cup C \cup S_1$,
2. H_2 is the union of the edges which join v_2 to the vertices of $A \cup C \cup S_2$,
3. H_3 is the union of the edges which join v_2 to the vertices of $\{v_1\} \cup A \cup S_2$.

Next, note that each of H_1, H_2, and H_3 is a star graph. Thus, $\mu(H_i) = 1$, for $i = 1, 2, 3$. So, if we define

$$\begin{aligned} X &= \{(i,j) : (v_i, v_j) \in \mathrm{V}(H_1)^2 \cup \mathrm{V}(H_2)^2 \cup \mathrm{V}(H_3)^2\} \\ &= \{(i,j) : (v_i, v_j), (v_j, v_i) \notin (S_1 \times S_2) \cup (S_1 \times A) \cup (S_2 \times B) \cup (A \times B)\}, \end{aligned}$$

by inequality (1), one can easily verify that

$$\begin{aligned}\sum_{\substack{(i,j)\in X\\ i<j}} (y_i-y_j)^2 \quad &\leq \quad (b+c+s+2) \sum_{\{i,j\}\in E(H_1)} (y_i-y_j)^2 \\ &\quad + (a+c+s+1) \sum_{\{i,j\}\in E(H_2)} (y_i-y_j)^2 \\ &\quad + (a+s+2) \sum_{\{i,j\}\in E(H_3)} (y_i-y_j)^2 \ . \end{aligned} \tag{4}$$

Since the resistance of a path with length 3 is equal to 3, we have

$$\begin{aligned}\sum_{\substack{v_i\in S_1\\ v_j\in S_2}} (y_i-y_j)^2 &\leq 3 \sum_{\substack{v_i\in S_1\\ v_j\in S_2}} \big((y_i-y_1)^2+(y_1-y_2)^2+(y_2-y_j)^2\big) \\ &= 3s \sum_{v_i\in S_1} (y_1-y_i)^2 + 3s \sum_{v_j\in S_2} (y_2-y_j)^2 + 3s^2(y_1-y_2)^2. \end{aligned} \tag{5}$$

Also, if we define $A_i := A \setminus N_{\overline{G}}(v_i)$, for each $v_i \in S_1$, then $|A_i| \leq l$, for each $v_i \in S_i$, and

$$\begin{aligned}\sum_{v_i\in S_1}\sum_{v_j\in A_i} (y_i-y_j)^2 &\leq 3 \sum_{v_i\in S_1}\sum_{v_j\in A_i} \big((y_i-y_1)^2+(y_1-y_2)^2+(y_2-y_j)^2\big) \\ &\leq 3sl\Big(\sum_{v_i\in S_1} (y_i-y_1)^2 + \sum_{v_j\in A} (y_2-y_j)^2 + (y_1-y_2)^2\Big). \end{aligned} \tag{6}$$

Similarly, if we define $B_i := B \setminus N_{\overline{G}}(v_i)$, for each $v_i \in S_2$, we have

$$\sum_{\substack{v_i\in S_2\\ v_j\in B_i}} (y_i-y_j)^2 \leq 3sl\Big(\sum_{v_i\in S_2} (y_i-y_2)^2 + \sum_{v_j\in B} (y_1-y_j)^2 + (y_1-y_2)^2\Big). \tag{7}$$

On the other hand, for all pairs (i,j) such that $v_i \in S_1$ and $v_j \in A \setminus A_i$, or $v_i \in S_2$ and $v_j \in B \setminus B_i$, or $v_i \in A$ and $v_j \in B$, we have the trivial inequality

$$(y_i-y_j)^2 \leq (y_i-y_j)^2. \tag{8}$$

Now, note that for all of the terms $(y_i-y_j)^2$ which appear in the right hand side of the inequalities (4),(5),(6),(7), and (8), we have $\{i,j\} \in E(\overline{G})$, and on the other hand, for each $i<j$, one of $(y_i-y_j)^2$ or $(y_j-y_i)^2$ appears in the left hand side of one of these inequalities. Since H_1 and H_2 have no common edges, $b+c+s+2 \geq a+c+s+1$, and the terms $(y_i-y_j)^2$ which appear in (8) does not appear in other inequalities, by summing up both sides of these inequalities, we

will get

$$\begin{aligned}\sum_{i<j}(y_i-y_j)^2 &\le \Big((b+c+s+2)+(a+s+2)+3s^2+3sl+3sl\Big)\\ &\quad\times\sum_{\{i,j\}\in E(\overline{G})}(y_i-y_j)^2\\ &= (n+2+3s^2+6sl)\sum_{\{i,j\}\in E(\overline{G})}(y_i-y_j)^2. \qquad \text{(by (2))} \quad (9)\end{aligned}$$

But $y=(y_1,\ldots,y_n)$ is an eigenvector of $L(\overline{G})$ corresponding to $\overline{\mu}$ with $\sum_i y_i = 0$, so have

$$\sum_{\{i,j\}\in E(\overline{G})}(y_i-y_j)^2 = y^T L(\overline{G})\, y = \overline{\mu}\|y\|^2 = \frac{\overline{\mu}}{n}\sum_{i<j}(y_i-y_j)^2.$$

Therefore

$$\overline{\mu} \ge \frac{n}{n+2+3s^2+6sl}. \qquad (10)$$

A lower bound for $\mu+\overline{\mu}$. Therefore, by inequalities (3) and (10),

$$\mu+\overline{\mu} \ge \frac{n}{n+2+3s^2+6sl} + \frac{s-1}{3} + \frac{l+1}{l+3}. \qquad (11)$$

Now by inequality (3), for $s \ge 3$, we have $\mu \ge 1$. So we can suppose that $s = 1$ or 2.

The case $s=1$. In this case,

$$\mu+\overline{\mu} \ge \frac{l+1}{l+3} + \frac{n}{n+5+6l}.$$

Thus, for each $n \ge 12$,

$$\mu+\overline{\mu} \ge 1 - \frac{2}{l+3} + \frac{12}{17+6l} > 1.$$

The case $s=2$. Similarly, in this case,

$$\mu+\overline{\mu} \ge \frac{1}{3} + \frac{l+1}{l+3} + \frac{n}{n+14+12l}.$$

Note that, when $l \ge 3$, $\mu \ge \frac{1}{3} + \frac{l+1}{l+3} \ge 1$. So, $\mu+\overline{\mu} > 1$. Also, for each of cases $l = 0,1,2$, when $n \ge 10$,

$$\begin{aligned} l=0: &\qquad \mu+\overline{\mu} \ge \frac{2}{3} + \frac{10}{24} > 1,\\ l=1: &\qquad \mu+\overline{\mu} \ge \frac{1}{3} + \frac{1}{2} + \frac{10}{36} > 1,\\ l=2: &\qquad \mu+\overline{\mu} \ge \frac{1}{3} + \frac{3}{5} + \frac{10}{48} > 1.\end{aligned}$$

For all of the remaining cases, the theorem can be checked numerically as follows:

- $n \leq 4$. The only connected graph G with connected complement, is P_4 the path graph of length 3. In this case:
$$\mu + \overline{\mu} = 2\mu(P_4) = 2(2 - \sqrt{2}) > 1.$$

- $s = 2, n \leq 9$. G_2 is a subgraph of $\overline{G}$ and so $\mu(G_2) \leq \overline{\mu}$. The graph G_2 only depends on the nonnegative integers a, b, c with constraints $s = 2$ and $a + b + c + 2s + 2 = n \leq 9$ (at most n^3 cases for each n). The minimum of the values of $\mu(G_2)$ in all of these cases is approximately 0.357, and so by (3) we have
$$\mu + \overline{\mu} > \frac{2}{3} + 0.35 > 1.$$

- $s = 1,\ 5 \leq n \leq 11$. In these cases also the values of the second Laplacian eigenvalues of all graphs G_2 can be computed numerically and the minimum is approximately 0.429. So again by (3), for $l \geq 2$ we have
$$\mu + \overline{\mu} > \frac{l+1}{l+3} + 0.42 > 1.$$

  For the remaining cases $l = 0, 1$, $|S_1| = |S_2| = s = 1$ and note that by the definition of l, the vertex in $S_1 = \{u\}$ is adjacent in G to at most l vertices in A, and also the vertex in $S_2 = \{v\}$ is adjacent in G to at most l vertices in B. Let G_3 be the graph obtained from G_2 by adding the edges in $\overline{G}$ between u and A and also edges between v and B.

  In the case $l = 0$, G_3 is only dependent (up to isomorphism) on the size of the sets A, B, C (at most n^3 cases for each n), and by computation for $5 \leq n \leq 11$ the minimum of $\mu(G_3)$ is approximately 0.697. Now G_3 is a subgraph of $\overline{G}$ and again by (3),
$$\mu + \overline{\mu} \geq \frac{1}{3} + \mu(G_3) > \frac{1}{3} + 0.69 > 1.$$

  In the case $l = 1$, u is adjacent in G_3 to all vertices in A except at most one vertex and also the same happens for v and B. By deletion of edges (if necessary) we can suppose that $A = \emptyset$ or u is nonadjacent in G_3 to exactly one vertex in A, and also the simillar thing for v with respect to B. Again the graph G_3 depends only (up to isomorphism) on the size of the sets A, B, C and for $n \leq 11$ the minimum of $\mu(G_3)$ is approximately 0.518. So we have
$$\mu + \overline{\mu} \geq \frac{l+1}{l+3} + \mu(G_3) > \frac{1}{2} + 0.51 > 1.$$

Thus, the theorem holds for every $n \geq 2$. □

3.2 Maximum of the algebraic connectivity of a graph and its compelement

As a by-product of the proof of Theorem 1, we prove the following theorem.

Theorem 2. *For all graphs G with n vertices,*

$$\max\{\mu(G), \mu(\overline{G})\} \geq 1 - O(n^{-\frac{1}{3}}).$$

Proof. It is sufficient to prove the theorem for a connected graph G with n vertices, for sufficiently large integers n. With the notations and assumptions at the beginning of the proof of Theorem 1, we know

$$\mu + \overline{\mu} \geq \frac{1}{(x_1 - x_2)^2}.$$

So, if $(x_1 - x_2)^2 \leq \frac{1}{2}$, then $\mu + \overline{\mu} \geq 2$ and the maximum of μ and $\overline{\mu}$ is at least 1. Thus we can suppose that $(x_1 - x_2)^2 \geq \frac{1}{2}$. Now, similar to Step 2 of the proof of Theorem 1, we can suppose that both μ and $\overline{\mu}$ are smaller than 1. This implies that the distance between v_1 and v_2 in G is equal to 3. Let $s \geq 1$ be the maximum number of vertex-disjoint paths with length 3 between two vertices v_1 and v_2 in G. Therefore,

$$1 > \mu \geq \frac{(x_1 - x_2)^2}{R^G_{1,2}} \geq \frac{1}{2} \times \frac{s}{3} = \frac{s}{6}.$$

So $s \leq 5$. Now, by (10), with the notations which was defined in Step 2.2 of the proof of Theorem 1, we have

$$\overline{\mu} \geq \frac{n}{n + 2 + 3s^2 + 6sl} \geq \frac{n}{n + 80 + 30l}.$$

On the other hand, note that according to the definition of l, G contains the subgraph G_1 which is illustrated in Figure 2. Without loss of generality, we can assume that the left vertex in Figure 2 is v_1 and the right vertex is v_2. Now we have

$$\mu x_1 = \sum_{\{1,i\} \in E(G)} (x_1 - x_i) \geq (x_1 - x_k).$$

So $x_k \geq (1 - \mu)x_1$ and $x_k - x_2 \geq (1 - \mu)(x_1 - x_2)$. Therefore,

$$1 > \mu = \sum_{\{i,j\} \in E(G)} (x_i - x_j)^2 \geq \frac{(x_k - x_2)^2}{R^G_{k,2}} \geq (1 - \mu)^2 \times \frac{1}{2} \times \frac{l + 1}{2}.$$

Thus, $(1 - \mu)^2 \leq \frac{4}{l+1}$ and $\mu \geq 1 - \frac{2}{\sqrt{l+1}}$. Now, we consider two cases:

1. $n < l^{\frac{3}{2}}$. Thus $n^{\frac{1}{3}} < l^{\frac{1}{2}}$ and

$$\mu \geq 1 - \frac{2}{\sqrt{l}} \geq 1 - \frac{2}{n^{\frac{1}{3}}}.$$

2. $n \geq l^{\frac{3}{2}}$. So $l \leq n^{\frac{2}{3}}$, $\frac{l}{n} \leq n^{-\frac{1}{3}}$, and

$$\overline{\mu} \geq 1 - \frac{80+30l}{n+80+30l} \geq 1 - \frac{80}{n} - \frac{30l}{n} \geq 1 - \frac{110}{n^{\frac{1}{3}}}.$$

Therefore, in all cases we have $\max\{\mu, \overline{\mu}\} \geq 1 - O(n^{-\frac{1}{3}})$. □

Remark. For each $n \geq 4$, define a graph G with vertices $\{v_1, \ldots, v_n\}$ such that the induced subgraph on $\{v_3, \ldots v_n\}$ is a complete graph, v_1 is only adjacent to v_3 and v_2 is only adjacent to v_4. One can observe that

$$\mu(G) = \mu(\overline{G}) = \frac{n - \sqrt{n^2 - 4n + 8}}{2} = 1 - \frac{1}{n} + O(\frac{1}{n^2}) < 1.$$

Therefore, for each $n \geq 4$, there exist a graph G which has n vertices and the maximum of $\mu(G)$ and $\mu(\overline{G})$ is less than 1.

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