Categorical Steenrod algebras

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Abstract

We define a certain type of subcategories of the category of graded differential r-algebras and associate to such subcategories a sequence of algebras $\mathcal{A}_n(R)$ ($n \geq 0$ and R a commutative ring). We show that the case $n = 2$ is the extension of the well known mod 2 Steenrod algebra $\mathcal{A}_2$. A brief calculation on Adem relations of the mod 4 Steenrod algebra are also given.

1 Introduction

The mod p Steenrod algebra $\mathcal{A}_p$ at the prime p is the algebra of stable cohomology operations

$$\theta : H^m(-, \mathbb{F}_p) \to H^n(-, \mathbb{F}_p)$$

under the composition $H^m(X; \mathbb{F}_p) \xrightarrow{\theta_X} H^n(X; \mathbb{F}_p) \xrightarrow{\theta'_X} H^k(X; \mathbb{F}_p)$. This algebra is completely characterized by the Steenrod squares Sq^k, for $p = 2$, and Steenrod reduced powers $\mathcal{P}^k$ accompanied by the Bockstein homomorphism involved the sequence

$$0 \to \mathbb{Z}/(p) \to \mathbb{Z}/(p^2) \to \mathbb{Z}/(p) \to 0,$$

for $p > 2$.

Let R be a commutative ring with unity. While the ring of stable cohomology operations $H^m(-, R) \to H^n(-, R)$ exist, it is hard to determine its structure. One of the difficulties is that the group of all cohomology operations $H^m(-, R) \to H^n(-, R)$ is isomorphic to the cohomology $H^n(K(R, m); R)$, where $K(R, m)$ is the Eilenberg-MacLane space (see eg. [1, 2, 3]).

The problem of computing the cohomology groups $H^n(K(R, m); R)$ is not completely solved. Thus, the study of ring of stable cohomology operations $H^m(-, R) \to H^n(-, R)$ is not so known.

There are many attempts of extending the Steenrod algebra $\mathcal{A}_2$ over an arbitrary commutative ring with unity R. The most significant of them is the algebraic introduction to the Steenrod algebra by Smith [5], who defines the Steenrod algebra $\mathcal{P}^*(\mathbb{F}_q)$ over any Galois field $\mathbb{F}_q$. Smith redefines the Steenrod reduced powers $\mathcal{P}^k$ as certain natural transformations of the functor $\mathbb{F}_q[-]$, which assign to any finite dimensional vector space V, its dual algebra $V^* =$

$\mathbb{F}_q[V]$ being simply a polynomial in $\dim_{\mathbb{F}_q} V$ indeterminates. Then he embeds $\mathcal{P}^*(\mathbb{F}_q)$ into the $\mathcal{P}^*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}_q$ as a Hopf subalgebra.

Another important extension of the Steenrod algebra is the Wood's integral Steenrod algebra $\mathcal{N}$ [7]. He uses differentiation as a property of the Steenrod square Sq^k, and defines the integral Steenrod squares as differential operators acting on $W = \mathbb{Z}[x_1, x_2, \ldots]$ by

$$SQ^k = \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} x_{i_1}^2 \cdots x_{i_k}^2 \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}},$$

where the summation runs over all sequences $(i_1, \ldots, i_k)$ of non-negative integers. Then he proves that $\mathcal{N} \otimes \mathbb{F}_2 \cong \mathcal{A}_2$, the mod 2 Steenrod algebra.

Both of the aforementioned extensions generalize Steenrod operations and then collect these operations inside a Hopf algebra.

In this paper, we extend the mod 2 Steenrod algebra $\mathcal{A}_2$ considering some properties of it. The main properties of $\mathcal{A}_2$ is the following, called the unstability properties [4].

1. If $x \in H^*(X; \mathbb{F}_2)$ and $k > |x|$, then $Sq^k(x) = 0$, where $|x|$ denotes the degree of x;

2. For any $x, y \in H^*(X; \mathbb{F}_2)$ and any $k \geq 0$,

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y);$$

3. For any $x \in H^*(X; \mathbb{F}_2)$, $Sq^{|x|}(x) = x^2$.

Through this paper, we assume R is a commutative ring with unity.

2 Unstable categories

Let $\mathsf{GrAlg}(R)$ denote the category of all graded commutative R-algebras with algebra homomorphisms as morphisms.

Definition 2.1. A subcategory $\mathcal{C}$ of $\mathsf{GrAlg}(R)$ is called unstable if there exist symbols $\{d^0, d^1, d^2, \ldots\}$ such that any object A of $\mathcal{C}$ is a left module over the non-commutative free R-algebra $S = R\langle d^0, d^1, \ldots\rangle$ provided the following properties hold.

1. d^0 acts identically on the objects of $\mathcal{C}(R)$; i.e., $d^0 = \mathrm{id}$.

2. For any object A of $\mathcal{C}$ and any $a \in A$, $d^k(a) = 0$ if $k > \deg(a)$;

3. (Cartan formula) Given any object A of $\mathcal{C}$ and any $a, b \in A$, any $k \geq 0$,

$$d^k(ab) = \sum_{i+j=k} d^i(a) d^j(b);$$

4. Given the objects $A, B \in \mathcal{C}$ and morphism $f : A \to B$, $f(d^k(a)) = d^k(f(a))$ for all $a \in A$ and all $k \geq 0$.

Alternatively, we write $\mathcal{C}(R)$ for $\mathcal{C}$ to emphasis the ring R.

The R-algebra $S = R\langle d^0, d^1, \ldots\rangle$ has a Hopf algebra structure given by

$$\Delta : S \to S \otimes_R S,$$
$$\Delta(d^k) = \sum_{i+j=k} d^i \otimes d^j.$$

The next result follows from this fact.

Proposition 2.2. *Let $\mathcal{C}$ be unstable over $S = R\langle d^0, d^1, \ldots\rangle$. Then $\mathcal{C}$ is closed under the tensor product.*

Proof. For objects A and B of $\mathcal{C}$, the map

$$\nu : (S \otimes_R S) \otimes (A \otimes_R B) \to A \otimes_R B$$

defined by

$$(d^i \otimes d^j)(a \otimes b) = (-1)^{j \deg(a)} d^i(a) \otimes d^j(b) \tag{1}$$

gives $A \otimes_R B$ a left module structure over $S \otimes_R S$. Now the composition

$$S \otimes_R (A \otimes_R B) \xrightarrow{\Delta \otimes \mathrm{id}} (S \otimes_R S) \otimes (A \otimes_R B) \xrightarrow{\nu} A \otimes_R B$$

makes $A \otimes_R B$ a left S-module.

For $a \otimes b \in A \otimes_R B$ and $k > \deg(a \otimes b) = \deg(a)\deg(b)$, from (1) we have by Cartan formula,

$$d^k(a \otimes b) = \sum_{i+j=k} d^i(a) \otimes d^j(b) = 0,$$

since either $k > \deg(a)$ or $k > \deg(b)$. Also,

$$\begin{aligned} d^k((a_1 \otimes b_1)(a_2 \otimes b_2)) &= (-1)^{\deg(b_1)\deg(a_2)} d^k(a_1 a_2 \otimes b_1 b_2) \\ &= (-1)^{\deg(b_1)\deg(a_2)} \sum_{i+j=k} d^i(a_1 a_2) \otimes d^j(b_1 b_2) \end{aligned}$$

□

Corollary 2.3. *There exist a topological space Z such that*

$$H^*(Z; \mathbb{F}_2) = H^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^*(Y; \mathbb{F}_2).$$

Proof. The category $\mathsf{Cohom}(\mathbb{F}_2)$ of cohomology rings of topological spaces is closed under the tensor product since it is unstable over $\mathbb{F}_2\langle Sq^0, Sq^1, \ldots\rangle$. Therefore, for the spaces X and Y, the tensor product $H^*(X;\mathbb{F}_2)\otimes_{\mathbb{F}_2} H^*(Y;\mathbb{F}_2)$ is an object of $\mathsf{Cohom}(\mathbb{F}_2)$. That is to say that, there exist a space Z such that

$$H^*(Z;\mathbb{F}_2) = H^*(X;\mathbb{F}_2)\otimes_{\mathbb{F}_2} H^*(Y;\mathbb{F}_2).$$

Of course, by Künneth formula, $Z = X \times Y$. □

Recall that a graded differential R-algebra is a graded R-algebra A together with a collection of linear maps $\{\partial_i : A \to A\}$ such that

$$\begin{aligned}\partial_i(ab) &= \partial_i(a)b + a\partial_i(b),\\ \partial_i(ra) &= r\partial_i(a), \partial_i\partial_j = \partial_j\partial_i,\end{aligned}$$

for all $a, b \in A$ and all $r \in R$. For any i, ∂_i is called a differential operator on A.

Consider the category $\mathsf{GrDiff}(R)$ of all graded differential R-algebras as objects. For the objects A and B of $\mathsf{GrDiff}(R)$, the morphism $f : A \to B$ is an algebra homomorphism preserving the differential operators; i.e., $f\partial_A = \partial_B f$, where ∂_A and ∂_B are differential operators on A and B, respectively.

Theorem 2.4. *Any unstable category $\mathcal{C}(R)$ is a subcategory of $\mathsf{GrDiff}(R)$.*

Proof. Suppose that $\mathcal{C}(R)$ is an unstable category over $S = R\langle d^0, d^1, \ldots\rangle$. We show that all objects of $\mathcal{C}(R)$ are differential algebras and all morphisms preserve differential operators. For any object A of $\mathcal{C}(R)$, from Cartan formula,

$$d^1(ab) = d^1(a)b + ad^1(b) \text{ and } d^1(r) = 0$$

for all $a, b \in A$ and all $r \in R$, since $\deg(r) = 0 < 1$. Therefore any object A of $\mathcal{C}(R)$ is a differential ring with differential operator d^1. Given a morphism $f : A \to B$ of objects $A, B \in \mathcal{C}(R)$, by property 3 of Definition 2.1, $fd^1 = d^1 f$ which shows f preserves the differential operator d^1. □

3 Steenrod algebra of an unstable category

One of the main concepts in the study of the Steenrod algebra is the ideal $\langle R_2(a,b) : a \le 2b\rangle$ of $\mathbb{F}_2\langle Sq^0, Sq^1, \ldots\rangle$, where

$$R_2(a,b) = Sq^a Sq^b - \sum_{j\ge 0} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

is called Adem relations with the property that for any cohomology class $\alpha \in H^*(X;\mathbb{F}_2)$, $R_2(a,b)(\alpha) = 0$.

In order to associate an algebra to an unstable category $\mathcal{C}(R)$, we need to have relations such that the object A of $\mathcal{C}(R)$ are 0 over them. We establish this relation in the following definition.

Definition 3.1. Let $\mathcal{C}(R)$ be an unstable category on $S = R\langle d^0, d^1, \ldots\rangle$. The Steenrod algebra associated to $\mathcal{C}(R)$, denoted by $\mathcal{A}_S(\mathcal{C}(R))$, is defined to be the quotient

$$R\langle d^0, d^1, \ldots\rangle / \bigcap_{A\in\mathcal{C}(R)} \mathrm{Ann}_S(A),$$

where $\mathrm{Ann}_S(A)$ is the annihilator of the S-module A. The ideal $\bigcap_{A\in\mathcal{C}(R)} \mathrm{Ann}_S(A)$ is called Adem relations of the category $\mathcal{C}(R)$.

For example, since $\mathsf{Cohom}(\mathbb{F}_2)$ is unstable over $\mathbb{F}_2\langle Sq^0, Sq^1, \ldots\rangle$, then we have $\mathcal{A}_S(\mathsf{Cohom}(\mathbb{F}_2)) = \mathcal{A}_2$, the mod 2 Steenrod algebra.

Let $\mathsf{GrDiff}_{-1}(R)$ be the category of graded differential algebras (A, ∂), where the differentiation operator ∂ decreases degree by 1. Morphisms in this category are algebra homomorphisms $f : A \to B$ preserving differential operators. For example, $R[x]$ is an object of $\mathsf{GrDiff}_{-1}(R)$. In the next result we exhibit a canonical Steenrod algebra and its Adem relations on category $\mathsf{GrDiff}_{-1}(R)$

Theorem 3.2. *The category* $\mathsf{GrDiff}_{-1}(R)$ *is unstable and its associated Steenrod algebra is the following divided polynomial algebra.*

$$\mathcal{A}_S(\mathsf{GrDiff}_{-1}(R)) = R\left\langle d_0^k : d_0^0 = 1,\, d_0^k d_0^l = \binom{k+l}{k} d_0^{k+l} \right\rangle.$$

Proof. Suppose that A is an object of $\mathsf{GrDiff}_{-1}(R)$ with differential operators $\{\partial_i\}$. Let $\partial(=\partial_A) = \sum_i \partial_i$. Put

$$d_0^0 = \mathrm{id},\, d_0^1 = \partial,\, d_0^k = \frac{1}{k!}\partial^k.$$

We claim that the category $\mathsf{GrDiff}_{-1}(R)$ is unstable over $S = R\langle d_0^k : k \geq 0\rangle$. For any $a \in A$ and any $k > \deg(a)$, $d_0^k(a) = 0$ since d_0^k decreases degree by k. From Leibnitz formula

$$\partial^k(ab) = \sum_{i+j=k} \binom{k}{i} \partial^i(a)\partial^j(b),$$

we have

$$d_0^k(ab) = \frac{1}{k!}\partial^k(ab) = \sum_{i+j=k} \frac{1}{i!j!}\partial^i(a)\partial^j(b) = \sum_{i+j=k} d_0^i(a) d_0^j(b).$$

Finally, for any morphism $f : A \to B$ in $\mathsf{GrDiff}_{-1}(R)$, $f(\partial_A(a)) = \partial_B(f(a))$ since f preserves each ∂_i. Thus, the first part of the theorem holds. To determine the associated Steenrod algebra $\mathcal{A}_S(\mathsf{GrDiff}_{-1}(R))$, we find out Adem relations as follows.

$$
\begin{aligned}
d_0^k d_0^l &= \frac{1}{k!}\frac{1}{l!} \sum_{(i_1,\ldots,i_k)} \sum_{(j_1,\ldots,j_l)} \partial_{i_1\ldots i_k}\partial_{j_1\ldots j_l} \\
&= \frac{1}{k!}\frac{1}{l!} \sum_{\substack{(i_1,\ldots,i_k)\\(j_1,\ldots,j_l)}} \partial_{i_1\ldots i_k j_1\ldots j_l} \\
&= \frac{(k+l)!}{k!l!} d_0^{k+l} \\
&= \binom{k+l}{k} d_0^{k+l}.
\end{aligned}
$$

This completes the proof. ☐

4 The n-Steenrod algebra

In this section, an unstable subcategory of $\mathsf{GrDiff}_{-1}(R)$ is introduced and a particular kind of Steenrod algebra is associated to it. Consider the subcategory $\mathsf{GrDiff}^1_{-1}(R)$ of $\mathsf{GrDiff}_{-1}(R)$ in which objects are all graded differential algebras A such that, for any differential operator ∂_i on A, there exists an element x_j of degree 1 in A such that

$$
\partial_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}
$$

It is clear that for the objects A, B and the morphism $f : A \to B$,

$$
\partial_i f(x_j) = f(\partial_i(x_j)) = \delta_{ij}.
$$

Being the subcategory of $\mathsf{GrDiff}_{-1}(R)$ with the same argument as in Theorem 3.2, the category $\mathsf{GrDiff}^1_{-1}(R)$ is unstable over $S_0 = R\langle d_0^k : k \geq 0\rangle$ and the associated Steenrod algebra $\mathcal{A}_{S_0}(\mathsf{GrDiff}^1_{-1}(R))$ is also a divided polynomial algebra. However, the aim of this section is to verify the unstability of $\mathsf{GrDiff}^1_{-1}(R)$ over an special algebra.

Theorem 4.1. *For any $n \geq 0$, there is a non-commutative free R-algebra $S_n = R\langle d_n^k : k \geq 0\rangle$ such that $\mathsf{GrDiff}^1_{-1}(R)$ is unstable on S_n.*

Proof. Given any object A of $\mathsf{GrDiff}^1_{-1}(R)$ with differential operators $\{\partial_i\}$ satisfying the $\partial_i(x_j) = \delta_{ij}$, define $d_n^0 = \mathrm{id}$, $d_n^1 = \sum_i x_i^n \partial_i$, and

$$
d_n^k = \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} x_{i_1}^n \cdots x_{i_k}^n \partial_{(i_1,\ldots,i_k)},
$$

where $\partial_{(i_1,\ldots,i_k)} = \partial_{i_1} \cdots \partial_{i_k}$. We claim that $\mathsf{GrDiff}^1_{-1}(R)$ is unstable over $S_n = R\langle d_n^k : k \geq 0\rangle$. The operator $\partial_{(i_1,\ldots,i_k)}$ decreases degree by k since ∂_i downs it by 1. Therefore for any $a \in A$ and any $k > \deg(a)$, we have $d_n^k(a) = 0$.

To prove Cartan formula for d_n^k, first note that

$$\partial_{(i_1,\ldots,i_k)}(ab) = \sum_S \partial_{(S)}(a)\partial_{(S')}(b),$$

where the summation is taken over all subsets $S = \{s_1, \ldots, s_r\}$ of $\{i_1, \ldots, i_k\}$, $1 \le r \le k$, S' is the complement of S in $\{i_1, \ldots, i_k\}$, and $\partial_{(S)}$ denotes the operator $\partial_{s_1} \cdots \partial_{s_r}$. Now we have

$$\begin{aligned}
d_n^k(ab) &= \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} x_{i_1}^n \cdots x_{i_k}^n \partial_{(i_1,\ldots,i_k)}(ab) \\
&= \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} x_{i_1}^n \cdots x_{i_k}^n \sum_S \partial_{(S)}(a)\partial_{(S')}(b) \\
&= \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} \sum_S x_{i_1}^n \cdots x_{i_k}^n \partial_{(S)}(a)\partial_{(S')}(b) \\
&= \frac{1}{k!} \sum_{(i_1,\ldots,i_k)} \sum_S \left(x_{(S)}^n \partial_{(S)}(a) \right) \left(x_{(S')}^n \partial_{(S')}(b) \right),
\end{aligned}$$

where $x_{(S)}^n = x_{s_1}^n \cdots x_{s_r}^n$. But, there exists $\binom{k}{r}$ subsets of $\{i_1, \ldots, i_k\}$ of size r, therefore, the Cartan formula holds, noting that for an arbitrary subset S of $\{i_1, \ldots, i_k\}$, of size $|S| = r$, we have

$$x_{i_1}^n \cdots x_{i_k}^n \partial_{(S)}(a)\partial_{(S')}(b) = \left(x_{(S)}^n \partial_{(S)}(a) \right) \left(x_{(S')}^n \partial_{(S')}(b) \right).$$

□

Definition 4.2. The associated Steenrod algebra $\mathcal{A}_{S_n}(\mathsf{GrDiff}^1_{-1})(R)$ is defined as the n-Steenrod R-algebra and denote by $\mathcal{A}_n(R)$.

Example 4.3. We experienced the 0-Steenrod R-algebra

$$\mathcal{A}_0(R) = R \left\langle d_0^k : d_0^k d_0^l = \binom{k+l}{k} d_0^{k+l} \right\rangle$$

in the preamble of Theorem 4.1, where the differential operators d_0^k, for $k \ge 0$, are as in Theorem 3.2. Another example is the commutative 1-Steenrod R-algebra $\mathcal{A}_1(R)$ with Adem relations

$$d_1^k d_1^l = \sum_{i=0}^{k} \binom{l}{i} \binom{k+l-i}{l} d_1^{k+l-i}.$$

The 1-Steenrod algebras are studied in details in [6].

5 Adem relations of the n-Steenrod algebras

For $n > 1$, computing Adem relations of the n-Steenrod algebra $\mathcal{A}_n(R)$ is not so easy. We attempt to some special cases.

Let $d_n = \sum_{i\geq 0} d_n^i$. Then $d_n(x_i) = x_i + x_i^n$. Therefore, for $q = p^v$ (p prime), the operations d_n^k in the algebra $\mathcal{A}_q(\mathbb{F}_q)$ are exactly the Steenrod reduced powers $\mathcal{P}^k$ with Adem relations as in the next result [5, section 2].

Theorem 5.1. *Suppose that $q = p^v$ is a power of the prime p. Let $\mathbb{F}_q$ be the Galois field of q elements. Then Adem relations of the q-Steenrod $\mathbb{F}_q$-algebra $\mathcal{A}_q(\mathbb{F}_q)$ are as follows.*

$$d_n^a d_n^b = \sum_{j=0}^{[a/q]} (-1)^{a-qj} \binom{(q-1)(b-j)-1}{a-qj} d_n^{a+b-j} d_n^j, \ \ a, b \geq 0, \ \ a \leq qb.$$

To compute Adem relations of n-Steenrod algebra $\mathcal{A}_n(R)$, we need some formulae from [7]. Define

$$D_n = \sum_i x_i^{n+1} \partial_i = d_{n+1}^1.$$

The commutative wedge product of $x_i^n \partial_i$ and $x_j^m \partial_j$ is defined by

$$x_i^n \partial_i \vee x_j^m \partial_j = x_i^n x_j^m \partial_{ij},$$

and is extended linearly to

$$\left(\sum_i x_i^n \partial_i\right) \vee \left(\sum_j x_j^m \partial_j\right) = \sum_{i,j} x_i^n x_j^m \partial_{ij}.$$

For example,

$$d_n^k = \frac{1}{k!} d_n^1 \vee d_n^1 \vee \cdots \vee d_n^1 = \frac{1}{k!} (d_n^1)^{\vee k}.$$

For a multiset $K = (n_1^{r_1} n_2^{r_2} \cdots n_a^{r_a})$, let

$$D(K) = \frac{D_{n_1}^{\vee r_1}}{r_1!} \vee \frac{D_{n_2}^{\vee r_2}}{r_2!} \vee \cdots \vee \frac{D_{n_a}^{\vee r_a}}{r_a!}.$$

For instance, $D(n) = D_n = d_{n+1}^1$ and $D(n^r) = \frac{1}{r!} D_n^{\vee r} = d_{n+1}^r$. Note that

$$D(n^{r_1} n^{r_2}) = D(n^{r_1}) \vee D(n^{r_2}) = \binom{r_1 + r_2}{r_1} D(n^{r_1+r_2}),$$

where $(n^{r_1} n^{r_2})$ and $(n^{r_1+r_2})$ are distinct multisets.

In his paper [7], Wood gives a general formula of decomposition $D(K) \circ D(L)$ for the multisets $K = (n_1^{r_1} n_2^{r_2} \cdots n_a^{r_a})$ and $L = (m_1^{s_1} m_2^{s_2} \cdots m_b^{s_b})$. For the present purpose, we need only the following two special cases. The first one is the following [7, Example 4.5].

Theorem 5.2. *We have*

$$\begin{aligned}D_n \circ (D_{m_1} \vee D_{m_2} \vee \cdots \vee D_{m_b}) = D_n &\vee D_{m_1} \vee D_{m_2} \vee \cdots \vee D_{m_b} \\ &+ (m_1+1) D_{n+m_1} \vee D_{m_2} \vee \cdots \vee D_{m_b} \\ &+ (m_2+1) D_{m_1} \vee D_{n+m_2} \vee \cdots \vee D_{m_b} + \cdots \\ &+ (m_b+1) D_{m_1} \vee D_{m_2} \vee \cdots \vee D_{n+m_b}.\end{aligned} \tag{2}$$

For example,

$$\begin{aligned}d^1_{n+1} d^k_{n+1} = D(n)D(n^k) &= \frac{1}{k!} D_n \circ (D_n \vee \cdots \vee D_n) \\ &= \frac{1}{k!}[D_n \vee D_n \vee \cdots \vee D_n + k(n+1) D_{2n} \vee D_n \vee \cdots \vee D_n] \\ &= \frac{1}{k!}[(k+1)! D(n^{k+1}) + (n+1) D((2n)^1) \vee D(n^{k-1})].\end{aligned}$$

Thus,

$$d^1_{n+1} d^k_{n+1} = (k+1) d^{k+1}_{n+1} + (n+1) D(n^{k-1}) \vee D((2n)^1). \tag{3}$$

The second special case is as follows [8, Example 2.10].

Theorem 5.3. *For multisets $N = (n^r)$ and $M = (m^s)$, we have*

$$d^r_{n+1} d^s_{m+1} = D(n^r) D(m^s) = \sum_{\Theta} \rho(\Theta) D(n^{\theta_0}) \vee \bigvee_{j=0}^{m+1} D((m+jn)^{t_j}), \tag{4}$$

where the summation is running over all solutions Θ of the simultaneous equations

$$r = \theta_0 + \sum_{i=1}^{n+1} i t_i, \quad s = t_0 + \sum_{i=1}^{n+1} t_i$$

in non-negative integers θ_0 and t_i and $\rho(\Theta) = \prod_{i=1}^{n+1} \binom{n+1}{i}^{t_i}$.

Now, we are ready to compute particular Adem relation of the n-Steenrod algebra for $n > 1$. We start with the grading 3.

Theorem 5.4. *In the n-Steenrod R-algebra $\mathcal{A}_n(R)$, we have*

$$3(n-1) d^3_n - (4n-2) d^2_n d^1_n + (n+1) d^1_n d^2_n + (n-1)(d^1_n)^3 = 0 \tag{5}$$

Proof. For the coefficients a_0, a_1, a_2, and $a_3 \in R$ put

$$a_0 d^3_n + a_1 d^2_n d^1_n + a_2 d^1_n d^2_n + a_3 (d^1_n)^3 = 0. \tag{6}$$

Using (3) and (4) we get

$$\begin{aligned}d^2_{n+1} d^1_{n+1} &= 3 d^3_{n+1} + (n+1) D(n^1) \vee D((2n)^1) + \binom{n+1}{2} D((3n)^1) \\ d^1_{n+1} d^2_{n+1} &= 3 d^3_{n+1} + (n+1) D(n^1) \vee D((2n)^1) \\ (d^1_{n+1})^3 &= 6 d^3_{n+1} + 3(n+1) D(n^1) \vee D((2n)^1) + (n+1)(2n+1) D((3n)^1).\end{aligned}$$

Substituting these equations in (6) gives

$$(a_0+3a_1+3a_2+6a_3)d^3_{n+1}+(n+1)(a_1+a_2+3a_3)D(n^1)\vee D((2n)^1)+ \left(\binom{n+1}{n}a_1+(n+1)(2n+1)a_3\right)(d^1_{n+1})^3=0$$

which leads to the homogenous linear system

$$\begin{bmatrix}1 & 3 & 3 & 6\\ 0 & 1 & 1 & 3\\ 0 & n & 0 & 4n+2\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}=0$$

of three equations and four unknowns a_0, a_1, a_2, and a_3. The result is now comes by solving this system of equations. □

Using Equations (3) and (4), we have

$$\begin{aligned}
d^1_{n+1}d^3_{n+1} &= 4d^4_{n+1}+(n+1)D(n^2)\vee D((2n)^1),\\
d^2_{n+1}d^2_{n+1} &= 6d^4_{n+1}+2(n+1)D(n^2)\vee D((2n)^1)+(n+1)^2D((2n)^2)\\
&+\binom{n+1}{2}D(n^1)\vee D((3n)^1),\\
d^3_{n+1}d^1_{n+1} &= 4d^4_{n+1}+(n+1)D(n^2)\vee D((2n)^1)\\
&+\binom{n+1}{2}D(n^1)\vee D((3n)^1)+\binom{n+1}{3}D((4n)^1),\\
d^1_{n+1}d^1_{n+1}d^2_{n+1} &= 12d^4_{n+1}+(n+1)D(n^2)\vee D((2n)^1)+2(n+1)^2D((2n)^2)\\
&+\quad (n+1)(2n+1)D(n^1)\vee D((3n)^1),\\
d^1_{n+1}d^2_{n+1}d^1_{n+1} &= 12d^4_{n+1}+5(n+1)D(n^2)\vee D((2n)^1)+2(n+1)^2D((2n)^2)\\
&+\frac{(n+1)(5n+2)}{2}D(n^1)\vee D((3n)^1)+(3n+1)\binom{n+1}{2}D((4n)^1),\\
d^2_{n+1}d^1_{n+1}d^1_{n+1} &= 12d^4_{n+1}+5(n+1)D(n^2)\vee D((2n)^1)+2(n+1)^2D((2n)^2)\\
&+(2n+1)(n+1)D(n^1)\vee D((3n)^1)+(n+1)\binom{n+1}{2}D((4n)^1),\\
(d^1_{n+1})^4 &= 24d^4_{n+1}+12(n+1)D(n^2)\vee D((2n)^1)+6(n+1)^2D((2n)^2)\\
&+4(n+1)(2n+1)D(n^1)\vee D((3n)^1)\\
&+(n+1)(2n+1)(3n+1)D((4n)^1).
\end{aligned}$$

Substituting the above equalities in the equation

$$a_0d^4_{n+1}+a_1d^3_{n+1}d^1_{n+1}+a_2d^2_{n+1}d^2_{n+1}+a_3d^1_{n+1}d^3_{n+1}+a_4d^1_{n+1}d^2_{n+1}d^1_{n+1}+ a_5d^1_{n+1}d^1_{n+1}d^2_{n+1}+a_6d^2_{n+1}d^1_{n+1}d^1_{n+1}+a_7(d_{n+1})^4=0. \quad (7)$$

leads to the homogenous linear system $AX = 0$ of five equations and eight unknowns, where

$$A = \begin{bmatrix} 1 & 4 & 6 & 4 & 12 & 12 & 12 & 24 \\ 0 & 1 & 1 & 1 & 5 & 1 & 5 & 12 \\ 0 & n & n & 0 & 5n+1 & 4n+2 & 4n+2 & 16n+8 \\ 0 & 0 & 1 & 0 & 2 & 2 & 2 & 6 \\ 0 & n(n-1) & 0 & 0 & 3n(3n+1) & 0 & 3n(n+1) & 6(2n+1)(3n+1) \end{bmatrix}.$$

By solving this system we get the next result.

Theorem 5.5. *In the n-Steenrod R-algebra $\mathcal{A}_n(R)$, the following equation holds.*

$$\begin{aligned} &4(n-1)[4(n+1)a_5 - 3(n-1)(5n-4)a_6 + 12(n+1)(2n-1)a_7]d_n^4 \\ &\quad + 3[2n(n-1)(3n-2)a_5 + 3n(n-1)^2a_6 - 2n(n+1)(3n-2)a_7]d_n^3 d_n^1 \\ &\quad + 2(n-1)[4(n-1)(2n-1)a_5 + (n-1)(5n-4)a_6 - 4(2n-1)(n+1)a_7]d_n^2 d_n^2 \\ &\quad + 2[-n(9n^2-4n+2)a_5 + 3(n-1)^2(n-2)a_6 - 6n(n-2)(2n-1)a_7]d_n^1 d_n^3 \\ &\quad + (n-1)[-2n(n-2)a_5 + n(n+1)a_6 + 2(13n^2-10n+4)a_7]d_n^1 d_n^2 d_n^1 \\ &\quad + 2(n-1)(3n^2-4n+2)a_5 d_n^2 d_n^1 d_n^1 + 2(n-1)(3n^2-4n+2)a_6 d_n^1 d_n^1 d_n^2 \\ &\quad + 2(n-1)(3n^2-4n+2)a_7 (d_n^1)^4 = 0 \end{aligned}$$

From equation (3), it is easy to see that in the algebra $\mathcal{A}_n(\mathbb{Z}/(n))$ we have

$$d_n^1 d_n^k = (k+1) d_n^{k+1}. \tag{8}$$

Also, we have

$$3d_n^3 - 2d_n^2 d_n^1 - d_n^1 d_n^2 + (d_n^1)^3 = 0,$$

and

$$\begin{aligned} 4[4a_5 - 12a_6 - 12a_7]d_n^4 + 2[4a_5 - 4a_6 + 4a_7]d_n^2 d_n^2 + 12a_6 d_n^1 d_n^3 \\ + 8a_7 d_n^1 d_n^2 d_n^1 + 4a_5 d_n^2 d_n^1 d_n^1 + 4a_6 d_n^1 d_n^1 d_n^2 + 4a_7 (d_n^1)^4 = 0. \end{aligned}$$

Noting the fact $d_n^1 d_n^2 = 3d_n^3$ and $d_n^1 d_n^1 = 2d_n^2$ in $\mathcal{A}_n(\mathbb{Z}/(n))$, we get the following equation in $\mathcal{A}_n(\mathbb{Z}/(n))$.

$$4(a_5 - 3a_6 - 3a_7)d_n^4 + 2(2a_5 + 3a_7)d_n^2 d_n^2 + 3a_6 d_n^1 d_n^3 + 6a_7 d_n^3 d_n^1 = 0.$$

6 4-Steenrod algebra mod 4

In this section we compute some Adem relations in $\mathcal{A}_4(\mathbb{Z}/(4))$. Recall from (4) that,

$$d_4^p d_4^q = D(3^p)D(3^q) = \sum_{\Theta} \rho(\Theta) D(3^{\theta_0}) \vee \bigvee_{j=0}^{4} D((3+3j)^{t_j}), \tag{9}$$

where the summation is taken over all non-negative integer solutions of the simultaneous equations

$$\begin{cases} \theta_0 + t_1 + 2t_2 + 3t_3 + 4t_4 = p, \\ t_0 + t_1 + t_2 + t_3 + t_4 = q, \end{cases} \tag{10}$$

and $\rho(\Theta) = \prod_{i=1}^{4} \binom{4}{i}^{t_i}$.

If Θ is a solution of (10) with $t_1 \neq 0$ or $t_3 \neq 0$, then $\rho(\Theta) = 0$, since $\binom{4}{1}^{t_1} = \binom{4}{3}^{t_3} = 0 \pmod 4$. This shows that the solutions with $t_1 \neq 0$ and $t_3 \neq 0$ do not effect (9). Also note that if $t_2 > 1$, then

$$\binom{4}{2}^{t_2} = 6^{t_2} = 2^{t_2} = 0 \pmod 4.$$

Therefore, in (10), only the solutions Θ with $t_1 = 0$, $t_2 = 0, 1$ and $t_3 = 0$ are valid. In other words, to compute $d_4^p d_4^q$ in $\mathcal{A}_4(\mathbb{Z}/(4))$, we need only to solve

$$\begin{cases} \theta_0 + 4t_4 = p \\ t_0 + t_4 = q \end{cases}, \tag{11}$$

and

$$\begin{cases} \theta_0 + 4t_4 = p - 2 \\ t_0 + t_4 = q - 1 \end{cases} \tag{12}$$

where in (11), t_4 ranges between 0 and $\min\{q, [p/4]\}$ and for any solution Θ of (11), $\rho(\Theta) = 1$. Also, in (12), $0 \leq t_4 \leq \min\{q-1, [(p-2)/4]\}$ and $\rho(\Theta) = 2$ for all solution Θ of (12). Therefore for non-negative integers p and q, in $\mathcal{A}_4(\mathbb{Z}/(4))$ we have

$$\begin{aligned} d_4^p d_4^q = & \sum_{j=0}^{\min\{q,[p/4]\}} \binom{p+q-5j}{p-4j} D(3^{p+q-5j}15^j) \\ & + \sum_{j=0}^{\min\{q-1,[(p-2)/4]\}} 2\binom{p+q-3-5j}{p-2-4j} d(3^{p+q-3-5j}9^1 15^j). \end{aligned} \tag{13}$$

Here, we used the abbreviation $D(n^{r_1}) \vee D(m^{r_2}) = D(n^{r_1} m^{r_2})$. In particular,

$$2d_4^p d_4^q = \sum_{j=0}^{\min\{q,[p/4]\}} 2\binom{p+q-5j}{p-4j} D(3^{p+q-5j}15^j). \tag{14}$$

One of the main concepts in Steenrod algebra, is the concept of admissibility. In $\mathcal{A}_p(\mathbb{F}_p)$ (p prime), the element $d_p^{a_1} d_p^{a_2} \cdots d_p^{a_r}$ is called admissible if $a_i \geq pa_{i+1}$ for $1 \leq i \leq r-1$. Adem relations on $\mathcal{A}_p(\mathbb{F}_p)$ provides a method to write the product $d_p^a d_p^b$ in terms of admissible monomials. In the sequel, we find a method to write $d_4^a d_4^b$ in terms of admissible monomials in a non-prime characteristic. As we will see, the result is not totally satisfactory because in a non-prime characteristic there are zero divisors which makes elements of $\mathcal{A}_4(\mathbb{Z}/(4))$ indecompossable.

For example, let compute $d_4^2 d_4^b$ and $d_4^3 d_4^b$. From (13), we have

$$d_4^2 d_4^b = \binom{b+2}{2} d_4^{b+2} + 2D(3^{b-1}) \vee D(9^1),$$
$$d_4^3 d_4^b = \binom{b+3}{3} d_4^{b+3} + 2bD(3^b) \vee D(9^1).$$

The term $2D(3^k) \vee D(9^1)$ appears only in these two products which makes $d_4^2 d_4^b$ indecomposible, while

$$2d_4^2 d_4^b = 2\binom{b+2}{2} d_4^{b+2} = \begin{cases} 2d_4^{b+2} & \text{if } b \equiv 0,1 \pmod 4 \\ 0 & \text{if } b \equiv 2,3 \pmod 4. \end{cases}$$

But $d_4^3 d_4^b$ may be decomposed since $2bD(3^b) \vee D(9^1) = bd_4^2 d_4^{b+1} - b\binom{b+3}{2} d_4^{b+3}$. In other words

$$d_4^3 d_4^b = \left[\binom{b+3}{3} - b\binom{b+3}{2}\right] d_4^{b+3} + bd_4^2 d_4^{b+1}.$$

Note also that

$$2d_4^3 d_4^q = 2\binom{q+3}{3} d_4^{q+3}.$$

Theorem 6.1. *For $p < 4q$, the monomial $2d_4^p d_4^q$ is a sum of admissible monomials, i.e. there are unique $B_k(p,q) \in \mathbb{Z}/(4)$ $(0 \le k \le [p/4])$ such that*

$$2d_4^p d_4^q = \sum_{k=0}^{[p/4]} 2B_k(p,q) d_4^{p+q-k} d_4^k. \tag{15}$$

Proof. When $p < 4q$, then $\min\{[p/4], q\} = [p/4]$ and $\min\{[(p-2)/4], q-1\} = [(p-2)/4]$. Also, for $0 \le k \le [p/4]$, $\min\{[(p+q-k)/4], k\} = k$. Therefore from (14), we have

$$2d_4^p d_4^q = \sum_{j=0}^{[p/4]} 2\binom{p+q-5j}{p-4j} D(3^{p+q-5j} 15^j),$$
$$2d_4^{p+q-k} d_4^k = \sum_{j=0}^{k} \binom{p+q-5j}{k-j} D(3^{p+q-5j} 15^j).$$

It means that the equation

$$2d_4^p d_4^q = \sum_{k=0}^{[p/4]} 2B_k(p,q) d_4^{p+q-k} d_4^k.$$

turns to

$$\sum_{j=0}^{[p/4]} 2\binom{p+q-5j}{p-4j} D(3^{p+q-5j}15^j)$$

$$= \sum_{k=0}^{[p/4]} \left[\sum_{j=0}^{k} 2B_k(a,b)\binom{p+q-5j}{k-j} D(3^{p+q-5j}15^j) \right]$$

$$= \sum_{j=0}^{[p/4]} \left[\sum_{k=j}^{[p/4]} 2B_k(p,q)\binom{p+q-5j}{k-j} \right] D(3^{p+q-5j}15^j).$$

The system

$$\binom{p+q-5j}{p-4j} = \sum_{k=j}^{[p/4]} B_k(p,q)\binom{p+q-5j}{k-j}$$

for $0 \leq j \leq [p/4]$ is a system of linear equation of the form $AX = B$ in unknowns $B_k(p,q)$ where the matrix A is so that all entries below main diagonal equal 0 and main diagonal entries equal to 1 because for any fixed j, the jth equation is

$$B_j + \sum_{k=j+1}^{[p/4]} \binom{p+q-5j}{k-j} B_k(p,q) = \binom{p+q-5j}{p-4j}.$$

Clearly, this system of linear equations has integer solutions. Reducing these solution modulo 4 gives the requested equation and uniqueness comes from the uniqueness of solution of the linear system. □

Corollary 6.2.

$$2B_k(4a+i,q) = 2\binom{q+i-k}{i} B(4a,q).$$

In particular,

$$2B_a(4a+i,q) = 2\binom{q-a+i}{i} = 2\binom{3(q-a)-1}{i}.$$

Proof. From the relations

$$2d_4^i d_4^b = 2\binom{b+i}{i} d_4^{b+i}, \quad (i = 1,2,3)$$

we have

$$2d_4^i d_4^{4a} = 2\binom{4a+i}{i} d_4^{4a+i} = 2d_4^{4a+i}.$$

It shows that

$$\begin{aligned}2d_4^{4a+i}d_4^q &= 2d_4^i d_4^{4a} d_4^q \\ &= \sum_{k=0}^{a} 2B_k(4a,q) d_4^i d_4^{4a+q-k} d_4^k \\ &= \sum_{k=0}^{a} 2B_k(4a,q)\binom{q+i-k}{i} d_4^{4a+i+q-k} d_4^k.\end{aligned}$$

Therefore the uniqueness of the coefficients B_k gives the requested. □

Example 6.3. For $i = 0, 1, 2$ and 3, we have

$$2d_4^{4+i}d_4^q = \sum_{j=0}^{[(4+i)/4]} 2\binom{3(q-j)-1}{4+i-4j} d_4^{4+i+q-j} d_4^j.$$

Also

$$2d_4^8 d_4^q = \sum_{j=0}^{[8/2]} 2\binom{3(q-j)-1}{8-4j} d_4^{q+8-j} d_4^j.$$

Now we investigate $d_4^p d_4^q$ for $p < 4q$. First let $p = 4$. Then from (13), we have

$$\begin{aligned}d_4^4 d_4^q &= \binom{4+q}{4} d_4^{4+q} + D(3^{q-1}15^1) + 2\binom{q+1}{2} D(3^{q+1}9^1), \\ d_4^{q+3} d_4^1 &= (q+4) d_4^{q+4} + D(3^{q-1}15^1) + 2D(3^{q+1}9^1).\end{aligned}$$

Therefore if $2\binom{q+1}{2} = 2 \pmod 4$, then $d_4^4 d_4^q$ may be written as sum of admissible monomials. But if $q = 0$ or $3 \pmod 4$ then $2\binom{q+1}{2} = 0 \pmod 4$ and in order to establish an equation for $d_4^4 d_4^q$ we need to add an extra non-admissible term, namely $d_4^2 d_4^{q+2} = \binom{4+q}{2} d_4^{4+q} + 2D(3^{q+1}9^1)$. The next theorem generalizes the calculation above.

Theorem 6.4. *For $p < 4q$, there are $R_k(p,q), S_k(p,q) \in \mathbb{Z}/(4)$ $(0 \le k \le [p/4])$ such that*

$$d_4^p d_4^q = \sum_{k=0}^{[p/4]} R_k(p,q) d_4^{p+q-k} d_4^k + \sum_{k=0}^{[p/4]} S_k(p,q) d_4^{2+4k} d_4^{p+q-2-4k}. \tag{16}$$

Proof. We look for elements $R_k(p,q)$ and $S_k(p,q)$ to establish (16). During the proof we write simply R_k and S_k instead of $R_k(p,q)$ and $S_k(p,q)$, respectively.

From (13), we have

$$\begin{aligned}
d_4^p d_4^q &= \sum_{j=0}^{[p/4]} \binom{p+q-5j}{p-4j} D(3^{p+q-5j}15^j) \\
&\quad + 2 \sum_{j=0}^{[(p-2)/4]} \binom{p+q-3-5j}{p-2-4j} D(3^{p+q-3-5j}9^1 15^j), \\
d_4^{p+q-k} d_4^k &= \sum_{j=0}^{k} \binom{p+q-5j}{p+q-k-4j} D(3^{p+q-5j}15^j) \\
&\quad + 2 \sum_{j=0}^{k-1} \binom{p+q-3-5j}{p+q-k-2-4j} D(3^{p+q-3-5j}9^1 15^j), \\
d_4^{2+4k} d_4^{p+q-2-4k} &= \sum_{j=0}^{k} \binom{p+q-5j}{2+4k-4j} D(3^{p+q-5j}15^j) \\
&\quad + 2 \sum_{j=0}^{k} \binom{p+q-3-5j}{4k-4j} D(3^{p+q-3-5j}9^1 15^j).
\end{aligned}$$

The right hand side of (16) then is

$$\begin{aligned}
&\sum_{k=0}^{[p/4]} R_k \left[\sum_{j=0}^{k} \binom{p+q-5j}{k-j} D(3^{p+q-5j}15^j) \right. \\
&\qquad\qquad \left. + 2 \sum_{j=0}^{k-1} \binom{p+q-3-5j}{k-j-1} D(3^{p+q-3-5j}9^1 15^j) \right] \\
&\quad + \sum_{k=0}^{[p/4]} S_k \left[\sum_{j=0}^{k} \binom{p+q-5j}{2+4k-4j} D(3^{p+q-5j}15^j) \right. \\
&\qquad\qquad \left. + 2 \sum_{j=0}^{k-1} \binom{p+q-3-5j}{4k-4j} D(3^{p+q-3-5j}9^1 15^j) \right] \\
&= \sum_{j=0}^{[p/4]} \left[\sum_{k=j}^{[p/4]} \binom{p+q-5j}{k-j} R_k + \binom{p+q-5j}{2+4k-4j} S_k \right] D(3^{p+q-5j}15^j) \\
&\quad +2 \sum_{j=0}^{[p/4]} \left[\sum_{k=j}^{[p/4]} \binom{p+q-3-5j}{k-j-1} R_k + \binom{p+q-3-5j}{4k-4j} S_k \right] D(3^{p+q-3-5j}9^1 15^j).
\end{aligned}$$

This shows that we have the system of linear equations

$$\begin{aligned}
&\sum_{k=j}^{[p/4]} \binom{p+q-5j}{k-j} R_k + \binom{p+q-5j}{2+4k-4j} S_k = \binom{p+q-5j}{p-4j} \\
&\sum_{k=j}^{[p/4]} \binom{p+q-3-5j}{k-j-1} R_k + \binom{p+q-3-5j}{4k-4j} S_k = \binom{p+q-3-5j}{p-2-4j}
\end{aligned} \tag{17}$$

for $0 \leq j \leq [p/4]$ which has integer solution. □

If $p = 4a + i$ ($i = 0, 1, 2$, or 3), then $[p/4] = a$ and from (17) we have

$$\begin{aligned}
S_a(4a+i, q) &= \binom{q-a+i-3}{i-2}, \\
R_a(4a+i, q) &= \binom{q-a+i}{i} - \binom{q-a+i}{2}\binom{q-a+i-3}{i-2} \\
&= \frac{i-2}{x+i-2}\binom{q-a+i}{i}\frac{2-(i+1)(q-a)}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
S_a(4a, q) &= 0, & S_a(4a+1, q) &= 0, \\
S_a(4a+2, q) &= 1, & S_a(4a+3, q) &= q-a.
\end{aligned}$$

As well,

$$\begin{aligned}
R_a(4a, q) &= 1, & R_a(4a+1, q) &= q-a+1, \\
R_a(4a+2, q) &= 0, & R_a(4a+3, q) &= (2q-2a+3)\binom{q-a+3}{2}.
\end{aligned}$$

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