A GENERAL FRAMEWORK FOR ERDOS-KO-RADO TYPE PROBLEMS

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Abstract. The famous Erdős-Ko-Rado theorem (EKR) states that when $2k < n$ are positive integers, the largest family of pairwise intersecting k-subsets of an n-set is when all the subsets share one specific element. Many different generalization of this theorem has been discovered. In this work, we propose a novel approach to prove EKR Theorem as well as some of its generalizations and related extremal combinatorics problems.

1. Introduction and Main Results

Erdős-Ko-Rado Theorem (EKR) states that for every pair $2k < n$ of positive integers, the maximum size of a family of pairwise intersecting k-subsets of an n set is equal to $\binom{n-1}{k-1}$ [2]. This is perhaps one of the most, if not the most, well-known problems concerning intersecting families. A general intersecting family problem asks the following question: An underlying set together with a given family of its subsets is given. What is the largest size of a sub-family such that every pair of elements intersect? When the family consists of all the k-subsets, then

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EKR answers the problem. A similar question can be asked when the underlying set is the edge set of a graph (For instance, the complete graph) and the family of the subsets is the set of all the edges of the graph with a certain property (for instance, all the matchings of a certain size). In this paper, we present a new approach that can prove the original EKR Theorem as well as many other questions of this type.

2. Main Results

In this section, we start by proving a simple lemma which is the basis for the consequent results. After stating it, we show how it can be used to conclude EKR-type results, including the original EKR Theorem.

Lemma 1. *Let A, X be two arbitrary finite sets and $\mathcal{R}$ be a family of subsets on X. If there is a map $f : A \times \mathcal{R} \to \{0, 1\}$ such that*

- *for every $r \in \mathcal{R}$, the number of $a \in A$ for which $f(a, r) = 1$ is at least ℓ and*
- *for every $a \in A$, the number of $r \in \mathcal{R}$ for which $f(a, r) = 1$ is at most t, then*

$$|\,\mathcal{R}\,| \leq \frac{t}{\ell}|A|.$$

Proof. Consider the number of pairs $(a, r) \in A \times \mathcal{R}$ with $f(a, r) = 1$. First, for a fixed a, the number of r's is at most t. Thus, this number is at most $t|A|$. Similarly, this number is at least $\ell|\,\mathcal{R}\,|$. Now, the assertion follows. □

Next, we derive several applications of the above Lemma. In fact, this lemma provides a unifying framework to prove such results.

Corollary 1 (EKR). *Let S be a set of size n and $\mathcal{U}$ be a family of k-subsets of it such that every pair of elements of $\mathcal{U}$ intersect. If $n > 2k$ then $|\mathcal{U}| \leq \binom{n-1}{k-1}$.*

Proof. In the above lemma set $\mathcal{R} := \mathcal{U}$ and define A to be the set of all cyclic permutations on $[n] = \{1, 2, \ldots, n\}$. Define the function f as follows. $f(a, r) = 1$ if and only if the permutation a preserves the elements of r cyclically consecutive. That is, the elements of r are located in positions s to $s + k - 1$ for some value of s and the locations are taken modulo n. One can observe that the conditions of the lemma holds for the values $\ell = n \cdot k! \cdot (n - k)!$ and $t = n$. Substituting to the assertion of the lemma and considering the fact that $|A| = n!$ we obtain that $|\,\mathcal{R}\,| \leq \binom{n-1}{k-1}$. □

The next application is the well-known Sperner Theorem which is regarding the maximum size anti-chain in the lattice of the subsets with inclusion order. An anti-chain is a set of elements such that no two of them are comparable.

Corollary 2. *Let n be a positive integer and S be a set of size n. Then, the largest size anti-chain of the subsets of S with inclusion relation is $\binom{n}{[\frac{n}{2}]}$.*

Proof. We will only sketch the proof due to the space constraint. In the lemma, let $\mathcal{R}$ be an anti-chain and A be the set of all permutations of $[n]$. Now, define the function f as follows. Let $f(a, r) = 1$ if and only if the elements of r appear at the beginning of the permutation a. It is not hard to observe that the parameters $t = 1$ and $\ell = \lceil \frac{n}{2} \rceil ! [\frac{n}{2}]!$ will fit the lemma. Thus, if we apply the lemma we conclude this corollary. □

The next two corollary can also be proved using our machinery. However, we leave their proofs because of the space limit.

Corollary 3. [5] *Let k be a positive integer number. For $n > 2k$, the maximum size subset of pairwise intersecting k-matchings of the graph K_n is when all these matchings share a particular edge in common.*

Corollary 4. *For any positive integer n, the maximum size of the pairwise intersecting family of the subsets of an n-set is 2^{n-1}.*

We now proof a new EKR-type result using the main lemma. We call a family of Hamiltonian cycles $\mathcal{H}$ of K_n is intersecting if any two cycles $C_1, C_2 \in \mathcal{H}$ has a common edge.

Theorem 1. *Let n be an odd positive integer. If $\mathcal{H}$ is an intersecting Hamiltonian cycles of K_n, then $|\,\mathcal{H}\,| \leq (n-2)!$; moreover equality holds if and only if $\mathcal{H}$ is the family of all Hamiltonian cycles of K_n that contain some fixed edge.*

Proof of Theorem 1. We prove the inequality. The second part of the theorem is not hard to show. First by using Lemma 1 we show that $|\,\mathcal{H}\,| \leq (n-2)!$.

Let S_n be a family of all permutation on $\{1, 2, \ldots, n\}$. Define

$$S_n^* = \{\pi |\, \pi \in S_n,\ \pi(1) = 1\}.$$

For a permutation $\sigma \in S_n^*$, define cycle D_σ as follows,

$$D_\sigma = v_{\sigma(1)} v_{\sigma(2)} \cdots \sigma(n)\sigma(1).$$

For a permutation $\sigma \in S_n^*$, define the permutation $\bar{\sigma} \in S_n^*$ as follows.

$$\forall i\ 2 \leq i \leq n\ \bar{\sigma}(i) = \sigma(n - i + 2)$$

For a Hamiltonian cycle C we assign two permutation $\pi_C, \bar{\pi}_C \in S_n^*$ such that $C = D_{\pi_C}$ and $C = D_{\bar{\pi}_C}$; moreover $\pi_{D_\sigma} \in \{\sigma, \bar{\sigma}\}$. Note that

For any permutation $\sigma \in S_n^*$ we have Hamiltonian cycles $D_\sigma, D_{\bar{\sigma}}$ are identical.

In Lemma 1, assume that $X = E(K_n)$, $A = S_n^*$ and $\mathcal{R}$ is a family of intersecting Hamiltonian cycles of K_n. In what follows we define $f : A \times \mathcal{R} \to \{0, 1\}$. By a theorem by Edouard Lucas in 1892, K_n has an edge decomposition to Hamiltonian cycles, say $\mathcal{C} = \{C_1, C_2, \cdots, C_{\frac{n-1}{2}}\}$(Note that ordering is important).

First without loss of generality assume that $\pi_{C_1} = id$. For any $\sigma \in A$ and $D \in \mathcal{R}$, define $f(\sigma, D) = 1$ if

$$D \in \{D_{\sigma\pi_{C_i}} | \, 1 \leq i \leq \frac{n-1}{2}\} \cup \{D_{\sigma\bar{\pi}_{C_i}} | \, 1 \leq i \leq \frac{n-1}{2}\},$$

else define $f(\sigma, D) = 0$.

Note that because $C_1, C_2, \cdots, C_{\frac{n-1}{2}}$ forms a decomposition of K_n all π_{C_i}'s and $\bar{\pi}_{C_i}$'s are distinct. Therefore, for each Hamiltonian cycle D we have exactly $n-1$ permutation in A such that $f(\sigma, D) = 1$.

Now, take a permutation σ in A, we shall show that only for one Hamiltonian cycle $v_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(1)}v_{\sigma(n)}v_{\sigma(1)}$, say D_σ, we have $f(\sigma, D_\sigma) = 1$. First note it is obvious that $f(\sigma, D_\sigma) = 1$. Now on the contrary, suppose that there are two intersecting Hamiltonian cycles D, D' for which $f(\sigma, D) = f(\sigma, D') = 1$. Therefore, without loss of generality assume that $D = D_{\sigma\pi_{C_2}}$ and $D' = D_{\sigma\pi_{C_3}}$

Assume that $ab \in E(D) \cap E(D')$. There for there are i, j such that $\sigma\pi_{C_2}(i) = \sigma\pi_{C_3}(j) = a$ and $\sigma\pi_{C_2}(i+1) = \sigma\pi_{C_3}(j+1) = b$. Therefore, $\pi_{C_2}(i) = \pi_{C_3}(i) = \sigma^{-1}(a)$ and $\pi_{C_2}(i+1) = \pi_{C_3}(j+1) = \sigma^{-1}(b)$, which means that C_2 and C_3 has an common edge. □

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