On the Hierarchy of Real Numbers

Arash Rastegar
Sharif University of Technology, Tehran, Iran
Institute for advanced study, Princeton, USA

We study the role of real numbers as the only structure which provides a hierarchy making measurement possible. We also motivate why functions are generalized numbers and study different patterns of growth in functions of one and several variables and compare differences between the concepts of hierarchy and norm on function spaces.

Introduction

The ring of integers $\mathbb{Z}$ is the soul of hierarchy of real numbers from which we construct rational numbers $\mathbb{Q}$ as the smallest field containing $\mathbb{Z}$ and the real numbers $\mathbb{R}$ as the smallest complete field containing $\mathbb{Q}$. The fact that $\mathbb{R}$ is complete makes it a tool appropriate for measurement. Indeed, $\mathbb{R}$ being the only totally ordered complete field, mathematically grasps what a model for a continuum should be.

When $\mathbb{R}$ is considered as a model for hierarchy, only $\mathbb{R}^{\geq 0}$ is relevant which is constructed from $\mathbb{N}\cup\{0\}$ and $\mathbb{Q}^{\geq 0}$ which are not groups with respect to addition. Although, the complete semiring $\mathbb{R}^{\geq 0}$ serves better as a height function on a manifold which is one of the incarnations of $\mathbb{R}^{\geq 0}$ as a tool for measurement, the complete field $\mathbb{R}$ which can be constructed from $\mathbb{R}^{\geq 0}$ by adding negatives, is a more simple mathematical structure which could be treated both algebraically and geometrically and arithmetically. More over considering solutions of equations, it is more natural to work with $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ rather than $\mathbb{N}\cup\{0\}$, $\mathbb{Q}^{\geq 0}$ and $\mathbb{R}^{\geq 0}$. Inequalities can be stated and proved more naturally if we work with whole of real number system. Considering height functions on lines , planes and space, we need a hierarchy which extends down to negative infinity in order to be able to model the fact that space and time are infinite in every direction. Although it seems philosophically sound to assume time and space are infinite only in the direction of future, thinking of infinite as an imaginable possibility in the future. Also modeling functions from $\mathbb{R}$ to $\mathbb{R}$ is simpler than modeling of functions from $\mathbb{R}^{\geq 0}$ to $\mathbb{R}^{\geq 0}$ specially working with polynomials and exponential functions. This is why we concentrate on $\mathbb{R}$ rather than $\mathbb{R}^{\geq 0}$.

Although, in some places we are forced to work with $\mathbb{R}^{\geq 0}$ like logarithm function. Distance from some point or some object is also always positive, and negative distance does not makes sense. Distance is also a model of hierarchy. Norm of an element of a vector space is also always a positive number. On the other hand structures over $\mathbb{R}$ can be easily generalized to structures over arbitrary field, but structures over $\mathbb{R}^{\geq 0}$ have no other analogue. For example, convex polytopes in a cone with center at origin are not a vector space but they are a module over the semi-ring $\mathbb{R}^{\geq 0}$. Distance from the origini , of the center of gravity of a convex polytope will serve as a nice norm on this module and also on vector space of all convex polytopes in $\mathbb{R}^n$.

Considering height functions on a line or plane or higher dimensional spaces, we need a hierarchy which extends down to negative infinity in order to be able to model the fact that space and time are infinite in every direction. Although, it seems philosophically sound to assume time and space are infinite only in the direction of future, thinking of infinite as an imaginable possibility in future. The author acknowledges support by an Oswald Veblen grant from Institute for Advanced Study.

1) Measurement

In another paper we explain why counting and measurement being discrete and continuous respectively should not be studied in the same mathematical model. This is why the generalizations of the concept of counting which lead to the notion of moduli space and universal objects are very much different from generalizations of the concept of measurement. (look at the closing section of [Ra1]).

There has not been much progem in the concept of measurement copared to the concept of counting because one does not consider the concept of evolution.One shall consider evolution of objects, concepts, structure, theorems, theories, language, and all higher levels of abstraction should be considered. For all these concepts of evolution, a corresponding concept of measurement should be introduced.

In physics, there is a widely accepted approach to the problem of measurement which we will try to adopt here. Quantities can be measured only when their measurement could be reduced to measurement of length. We shall keep this philosophical approach to the problem of measurement and also add a new principle which says that measurement is always relative; meaning that, we work with change in a parameter divided by the parameter itself. This way we have kept the concept of geometric measurement and adopted it to fit problems of higher continuum.

2) Geometric measurement revisited

Some phisicist believe that $\mathbb{Q}$ suffices for all practical measurement purposes. Mathematically this was soon proved to be wrong by discovery of irrational numbers. This lead to the idea of algebraic numbers and arithmetic of $\overline{\mathbb{Q}}$. This brings us to the real of $\overline{\mathbb{Q}}\cap\mathbb{R}$ or even totally real number fields. Many theorems in arithmetic which are true over $\mathbb{Q}$ could be generalized to totally real number fields. An important example being the fact that elliptic curves over $\mathbb{Q}$ are modular, which is one of the most advanced topics in the arithmetic in our time.

This direction ties geometric measurement to arithmetic. But the womb of history gave birth to existence of real numbers which are not algebraic. Even a counting argument could reveal the existence of such numbers. Although, explicit transcendental numbers are proved to exist. Proving that a number is not algebraic need a type of arithmetic which is not in the direction of geometric measurement.

3) Height functions, norms and valuations

The concept of height function and and norm of a vector space are closely tied to the geometric measurement of space. In fact if one considers DesCartes analytic geometry with clear mind, it is the sent of n independent height functions on $\mathbb{R}^n$. The concept of a norm on a vector space which models the concept of distance from a given point, is more closely related to the model $\mathbb{R}^{\geq 0}$ rather than all the real numbers. The concept of valuation on fields is closely related to norm but not related to geometric approach to measurement.

In fact the concept of distance has the capacity to be modeled without any reference to hierarchy. For example the distance of directions of two lines in a plane would be given by points on a circle. And distance between two lines in three dimensional space can be given by points on a projective space. The concept of how two lines are close to each other can be successfully modeled this way but not the concept of how far two lines are from each other which needs global measurement. So, philosophically if we want to define hierarchy by considering the distance of points or objects it has complicacies to be generalized.

What about height functions on a vector space? Meaning that a norm which could have negative values and points of norm zero form a sub-vector space. What about a number of height functions which have origin as the only point of the zero height sub-spaces? For example consider height functions on $\mathbb{R}^{\mathbb{Z}}$ or $\mathbb{R}^{\mathbb{R}}$. Then treating $\mathbb{Z}^{\mathbb{Z}}\subset\mathbb{R}^{\mathbb{Z}}$ or $\mathbb{Z}^{\mathbb{R}}\subset\mathbb{R}^{\mathbb{R}}$ as a discrete module on a base to do arithmetic in this huge vector space.

There are a number of objects associated to distance from a point which should be irrelevant when working with height functions with possible negative values. For example the concepts of center, circle, or more generally conic sections, sphere and other higher dimensional analogues. There are a number of objects associated to height function like point, line, plane and higher dimensional analogues Also zeroes of a single polynomial function and its generalizations.

This brings us to the question if $\mathbb{C}$ values or values in another field could be thought as a height function? At the moment I think probably not, since there are only norms on field which are $\mathbb{R}^{\geq 0}$ valued and not $\mathbb{R}$ valued. Another possible direction would be where ever we have intersection theory, there is expectation there to be possible to build a model of hierarchy also, and this is what we have for polynomial height functions on $\mathbb{C}^n$.

4) A new model for real numbers

There is a reconstruction of a model of real numbers introduced in [Ra1]. Assuming that M is a number which is so big that can not be imagined and e a number unimaginably small and also assuming that $eM=1$, we constructed a model of real numbers where each real number is of the form

$$\sum_{r\in\mathbb{R}} a_r M^r$$

in which a_r are coefficients in real numbers. Sum of such real numbers is defined component by component and product of such numbers is defined as polynomial like terms in terms of M . We could assume that a_r is zero except for finitely many of them so that the product be well-defined.

If we ignore the coefficients a_r we get something very similar to tropical semi-ring. But we shall not choose this path. Here, we see why functions from $\mathbb{R}$ to $\mathbb{R}$ are generalizations of real numbers and therefore could be considered as values of a height function. This will be our model to deal with the problem of evolution of objects and concepts and etc. Height of an object will not be of the form

$$\sum_{r\in\mathbb{R}} a_r M^r$$

but of the form $a_r M^r$ for some r.

If we do not want to keep the assumption that the new model of real numbers be totally ordered we can use the model of real numbers of the form

$$\sum_{r_I\in\mathbb{Z}} a_{r_I} M_I^{r_I}$$

where $M_I^{r_I}$ is of the form $M_1^{r_1} M_2^{r_2} \ldots M_k^{r_k}$. Here I is an index set which could be finite or infinite. This is very similar to laurent polynomial in several variables, and if the index set is finite , the function field is another interpretation for such numbers, namely

$$\mathbb{R}(X_{1,\ldots}, X_k)$$

One could of course consider infinitely generated fields over $\mathbb{R}$. This is very similar to the Indian origin of the concept of function by Bhaskara.

5) Asymptotic as heights

The function spaces $\mathbb{R}^{\mathbb{Z}} \subset \mathbb{R}^{\mathbb{R}}$ both could be modeled as generalizations of the set of real numbers. Therefore, they deserve to have a concept of height. We know them as reals of the form

$$\sum_{r\in\mathbb{Z}} a_r M^r \quad \text{and} \quad \sum_{r\in\mathbb{R}} a_r M^r$$

Such a function is usually compared to

$$\sum rM^r \quad \text{or} \quad \sum r^k M^r \quad \text{or} \quad \sum e^r M^r \quad \text{or} \quad \sum \log(r) M^r$$

which in the old language where represented by

$$n \text{ or } n^k \text{ or } e^n \text{ or } \log(n) .$$

Growth and decay should be both under focus. A function could have logarithmic, polynomial,or exponential growth or decay and this is a measure or a height on functions as if we have some sort of order on $\mathbb{R}^{\mathbb{R}}$ as if

$$e^{-n} < ... < n^{-k} < ... < n^{-2} < n^{-1} < 1/\log(n) < \log(n) < n < n^2 < ... < n^k < ... < e^n$$

We say $f \prec g$ if $\lim_{x\to\infty} f(x)/g(x) = 0$ and this will be a sort of order on functions. In case $\lim_{x\to\infty} f(x)/g(x)$ exists and is nonzero, we say f and g are comparable and we write $f \sim g$. We could limit ourselves to functions in $\mathbb{R}^{\mathbb{R}}$ that have finite or infinite limit. This subset will be also closed under addition and multiplication and more or less division. We could allow discrete zero sets and blow ups at points in $\mathbb{R}$ so that we have a closed set under division also., or we could assume that left or right limits everywhere exists and can be finite or infinite. The hierarchy above suggest that there could be a number N that is greater than all powers of unimaginable number M just like $e^n > n^k$ for any given power k. Then a typical real number would be of the form

$$\sum_{r \in \mathbb{R}} a_r M^r + \sum_{s \in \mathbb{R}} b_s N^s$$

which represent functions on $\mathbb{R}^2$ when we put dictionary order on $\mathbb{R}^2$. Similarly we could speak of functions on $\mathbb{R}^n$.

What would be the natural order on the space of functions on $\mathbb{R}^2$ or even $\mathbb{R}^n$ when the function could be decaying in one direction and growing in another direction? Could it be that the assumption $N > M$ giving a dictionary order on $\mathbb{R}^2$ may give us a solution to this puzzle?

References

[Ra1] Rastegar, Arash:"The problem of continuum Revisited”.