On The Dialogue between Geometry and Algebra

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Abstract: Geometry and algebra are two parallel rivers in mathematics, which flow each one leaning on the other. They are related to pictorial and verbal cognitive styles. This leads to the belief that any piece of Mathematics has parallel geometric and algebraic formulations. In this paper, we study the dialogue between these two approaches in mathematics.

Introduction

Through the course of history, algebra took more time to be formulated with respect to geometry. This means that algebraic thinking is more abstract and less available, and geometric thinking is more intuitive and more available. Despite this fact, the scope of intuitive thinking is much wider than abstract thinking. This means that there are parts of geometry, which are more abstract than all the scope of abstraction of algebra. I believe that spectrum of intuition contains the spectrum of algebra, or else you have to consider them as two separate realms, which are not directly correlated. We will study the dialogue through the course of history of Mathematics, until we reach the 20th century the different branches arise independently.

§1. Geometric Solutions of algebraic equations in Greek mathematics

Geometric Solution using ruler and compass is the way of Greeks to solve algebraic equations. But they had to deal with positive numbers and they had to consider different formulations for linear and quadratic equations. But this was as far as they could go. The concept of algebraic Solution of algebraic equations came long after being introduced by Khwarizmi. Even algebraic symbols came much after, they to write an equation in words not in mathematical symbols. Ruler and Compass were not the only devices to make accurate drawings but under the influence of elements and actually much before that, these tools became much more popular than others. There are devices to draw ellipses and parabolas and hyperbolas, graded and not graded and finite and infinite ruler, and Compass of bounded radius, constant radius and unbounded radius. Negative solutions didn't make sence before Descartes 16th century. At the Greek period, algebra was there but buried under of algebraic details which are not necessary. Notations for numbers and arithmetic were too far from the concept of notation developed for algebra much later. Even the concept of number was geometric and representation of numbers came much after that from India.

§2. Unity of concepts of number and shape

The concept of triangular numbers and square numbers are still widely used. They were invented by Pythagoreans to give a geometric taste to numbers. In fact, numbers and number sequences were still geometric concepts. They had to have a geometric pattern. I have not seen the concept of pyramid numbers and cubic numbers in the old literature but I am sure they were present. Higher dimensional patterns and patterns that could not stabilize on the desk were naturally not considered. Even today geometric patterns in sequences are used in teaching elementary mathematics and pictorial proof of identities are introduced. This indicates that originally there was a unity between the concept of number and shape. At least natural numbers and patterns were considered identical. This is how the concept of divisor and prime number came to become important so early, because it had geometric interpretation. Prime number are those which cannot be arranged in the form of a rectangle. People being interested in Pythagorean triple can be another reason for unity of the concepts of number and shape. When rational numbers were widely used and Diophantine problems with rational solutions became important and many of them were related to geometric problems and geometric patterns and shapes.

§3. Descartes following Khayyam and rise of analytic geometry

Greeks were not able to solve cubic equations with ruler and Compass. Khayyam used parabolas to solve them and in 17th century it was found that his method also worked for degree four equations. Khayyam was also the first person to use real line as a model for time and continuous movment and paved the way for Descartes for inventing analytic geometry in 16th century. This was a revolution in the dialogue between geometry and algebra. All the geometry on that moment of history was translated to algebraic language by Descartes. This brought us the idea, for the first time, that may be geometric thinking and algebraic thinking are two parallel languages. May be number and shape are incarnations of the same truth. May be one could do mathematics beyond these languages. Algebraization of geometry was a program proposed by Descartes to check if the wisdom could formulate all that comes to mind. This was a great victory for wisdom and the scale is tipped from 16th century towards verbal achievements in human civilization. That was a victory of Aristotle to Plato. Well! Many people think Aristotle defeated Plato much much earlier. But not in mathematics, certainly not in Archimedes' of infinites. Which was conquered even after Descartes in 19th century. We will discuss this in detail.

§4. From Galois to Gödel

Another victory of verbal thinking to intuitive geometry was the impossibility of solving the degree five equations and above by ruler and Compass. In 19th century several other attacks to intuition were formed. Introducing models for hyperbolic geometry by Klein and regorizing formulation of calculus by the concept of limit. Rise of set theory and Cantor's heaven for dealing with the concept of Infinity. At the same time there were many new lands discovered by geometers yet to be conquered by algebraists. From Newton and Leibniz to Euler and Lagrange intuitive mathematics florished, and it was Gödel finally who indicated that (only indicated that) the whole mathematics may not be axiomatizable. But definitely, in 19th century mathematicians raised the standards of and a huge volume of Euler's publication were driven to the valley of vanity. Intuitive mathematics was followed by mathematical in 20th century and indeed, it had a lot of achievements. May geometers and topologies from Riemann and Poincare to Thurston were criticized for their style of doing geometry. Let us concentrate now on the details of the dialogue between geometry and algebra is 20th century. What happened in this period is the holistic points of views we discussed above.

§5. Lie-groups and Lie-algebras

Lie groups are geometric objects but accommodated with algebraic structures, rather than being algebraic structures, which also accommodate geometric structure. The whole structure can be algebraically interpreted as Lie-algebras. Lie groups and Lie algebras because of their algebraic structure should be considered as part of parabolic mathematics. Representation of Lie groups and algebras can be thought as an analogue of embedding of manifolds in affine spaces. A finite group and its representations are also regarded as geometric objects, which encode symmetry of some mathematical structure. Representation of groups in GL(n) is also an analogue of embedding problem. Geometric objects should be deformed and studied if they are not rigid. For example, elliptic curves and abelian varieties should be studied in families. Lie groups tell us that some geometric objects could carry algebraic structure, but we expect that all geometric objects be modeled among this particular subset. Jacobian variety of Riemann surfaces and intermediate Jacobians of algebraic varieties are examples of this phenomena. There must be a way to define intermediate semi-abelian varieties for non-proper algebraic varieties. This raises the question that if not-proper algebraic varieties can be thought of proper ones in the limit, in such a way that intermediate semi-abelian variety tends to intimidate Jacobian.

§6. Algebraic groups and symmetric spaces

Algebraic groups and symmetric spaces are deeply related to modular forms which are arithmetic objects. This brings us to the philosophy that algebraic groups, which can be defined over arbitrary field, are deeply arithmetic or more generally algebraic objects and symmetric spaces accommodating their structure in a geometric object standing on the shoulders of arithmetic. This is another confirmation for the philosophy that number and shape are of the same origin or incarnating from the same truth. Hecke operators acting on modular form are deeply arithmetic structures. Many correspondences of symmetric spaces are of arithmetic origin. Algebraic groups are flexible to be formulated in many different languages, like Adelic language and Hopf algebras. Not all these formulations show all features. They are particular symmetric spaces, which have quotients, which are algebraic varieties or even moduli space of mathematical structures. Particular examples are moduli spaces, Shimura varieties, and modular curves, which are all connected to each other. They have models over integers with some prime numbers inverted, which confirms the arithmetic nature of algebraic groups.

§7. Homology and cohomology

Homology theory is the modern confirmation that number and shape are from the same origin and should be treated with the same mathematical language. All homology classes could correspond to shapes embedding in the space and all cohomology classes could correspond to differential forms. The differential operator d is analogue of the boundary operator in homology. The nice feature is that this analogy goes through for differential of product of functions. Indeed, there is an analogue of Leibniz formula for boundary operator. This shows that working with manifolds with boundary and even non-smooth boundaries is natural. Manifolds that are locally product of manifolds with boundary are such spaces. Such objects embedded in ambient manifolds should also be unified with numbers. This may be the solution to the longstanding paradox, which was discovered by Khayyam. Namely, why is it that product of two numbers on the real line is another number, but product of two intervals is a rectangle. This was the first instance of separation between concepts of number and shape. Khayyam Suggested that number could be unified with the concept of ratio, which in his views solves the paradox of dimension of shapes. Or we could assume that numbers have dimensions!

§8. Algebraic geometry

Algebraic geometry was originated over IR by Descartes and over by Abel. It was Hilbert who initiated the translation to the language of commutative algebra, and it was Emy Noether who initiated study of commutative rings in the abstract sense of the word, origin of all these objects come from intersection theory and from quadratic forms, and the language of algebra prepared the ground for generalization. But it all started from geometric intuition. Italian school introduced the intuitive path and Weil introduced the function field formulation and Grothendieck introduced the scheme formulation after formation of sheaves by Serre. The concept of geometric space become more and more abstract under the influence of algebra. One cannot say that more abstract means intuition. Scheme theory and sheaf theory are both quite intuitive. But proving theorem relies on the connections with algebra. This was why the intuitive style of Italian school was soon forgotten. The ability to give explicit proofs made Grothendieck's scheme theory more and more useful and links with arithmetic deepened through time and algebraic geometry started to have contributions to arithmetic geometry and eventually led to the birth of Arakelov theory.

§9. Non-commutative geometry

There are two main approaches to non-commutative geometry. One is the Connes style, which is based on functional analysis, and the other is Rosenberg style, which is based on non-commutative scheme, which relies on the category of query-coherent sheaves on a space. Here, they try to apply geometric intuition on objects, which are algebraic by pushing intuition too far. The realm of category theory is very flexible, therefore it is sort of possible to make such a push. Non-commutative geometry is where geometry lays on algebra versus algebraic geometry where algebra lays on geometry. Morphism between objects are of pure algebraic nature is non-commutative geometry. But in algebraic geometry, all geometric and intuitive ideas are translated and reformulated in the language of algebra. The concept of non-commutative trace by H.Bass can lead to a theory of non-commutative intersection theory, which is a huge gift to the world of intuition. Connes has some ideas relating Riemann hypothesis to trace of an operator on a non-commutative geometry space. It is safe to say that, in the dialogue between geometry and algebras there are places where we called them geometry where pure intuition is reduced to similarly with geometric structure, as we intuitively understand. This opens a discussion about analogy being part of intuition or a separate cognitive skill. To me it seems that analogy could be both geometric or algebraic and it should be treated as a different concept.

§10. Universal objects and universal properties

Universal objects and universal properties make sense for both algebraic objects and geometric objects. This is became of the “universality” of the concept of morphism, which is present in both ways of thinking. This means that universal properties can be thought of a tool for unifying the concept of number and shape. Unfortunately, morphism are between numerical structures not numbers and between spaces not shapes. The concept of function is in a way a morphism of numbers, especially if it is based on formulas, like a polynomial function. But it is in no way intrinsic. Maybe in the same way, a morphism between a number and a divisor could be imagined. Similarly, morphism between objects, which are geometric, are comprehendible. Like a morphism from quadrangle to a triangle or from a cube to a tetrahedron or from a rectangle to a cube or from a tetrahedron to a regular tetrahedron. But such concepts of morphism are not under focus in advanced mathematics. Morphism is treated as a global operator, which decides to do something on every object living in a world. But an operator accepting several possible options acting on an object is not treated on a concept of morphism.

§11. Function fields and number fields

One of the important collaborations between geometric thought and algebraic thinking was algebraic formulation of theory of Compact Riemann surfaces, which led to the analogous theory of number fields. Number fields have features that are not present in the function field. This is related to the concept of ordered field. This is linked heavily to the concept of growth in everyday life. So, there must be a truth in this feature. Question is if this concept has affected the theory of function fields. In fact, one can define the concept of height for all finitely generated fields, which are arithmetic objects. This has been a technique in proving results in Diophantine geometry. Also, there is the concept of norm which is absolutely essential is many many places in mathematics. Norms always have values in R. The concept of Aichimidus principle always appears in proofs of theorems and class field theory as was mentioned to me by C Skinner.

§12. Homotopy theory

Homotopy theory has a very geometric and intuitive foundations and it is actually very much related to continuity and order in R. But the forefront of the theory and its achievements are very algebraic. Many believe that the first achievements of category theory happened in homotopy theory. The algebraic nature make it appropriate to be mixed and related to scheme theory and that would be foundations for work of Voevodsky. Also, one can make a category theoretic foundation for homotopy theory, which is quite formal and independent of the idea of continuous change. This has a serious implication on the nature of truth and that is formal similarity of two theories make room for possibility of joining the two theories. This is not the same if two theories have the same essence. For example, Atiyah-Singer theorem and Grothendieck-Riemann-Roch have the same essence but their foundations are not compatible. This is why it is very difficult to join the two theories. May be reducing down to the language of formalism is a prerequisite for joining the essence of two independent jewels of truth. Same goes for homotopy theory and scheme theory. Voevodsky tried also to use homotopy theory for a reformulation of foundations of mathematics, which would fit for computer-aided proofs.

§13. Category theory

Category theory is formal. One can treat a category as the category of quasi-coherent sheaves on a geometrical space, and try to apply intuition. The difference between category theory and algebra is global treatment of mathematical objects. The same difference could make sense in geometric thinking. On can deal with geometric objects globally treating them as spaces and also deal with them locally treating them as shapes. Parallel to these is the idea of number against a numerical structure. But a morphism between algebras deal with destination of numbers one by one and a morphism in category theory is about global treatment of algebraic or geometric structures. Categorical treatment is always formal. Is there a global geometric understanding of geometric objects, which is not formal? In my opinion, this is pure intuition and pure observation of all related sub objects, which could be observed. Here we make a distinction between observable and non-observable geometric objects. For example, a continuous function is an observable object but a measure is not an observable object. If one has infinite time and energy, one can observe all geodesics, counts all solutions, and study their intersections and their correlations. This kind of mathematics is accessible to intuition. Category theory is formal and therefore not observable except of we think of it as a form of directed graph.

§14. Differential topology

Differential topology is about geometric objects and geometric intuition but the tools are analytic and local. Therefore computations are algebraic and formal but the result of computations is intuitive and geometric. Although one could conclude global deductions about the geometric objects by these global considerations are result of superposition of local manipulations. This is because differentiability is a local condition and that some of the results are surprisingly global like 28 differential structure on the sphere or infinitely many differential structures on four dimensional affine space. The method for distinguishing global properties is global invariants. The method of invariants works by associating a number or a structure or an elements in a structure associated to the problem which remains unchanged if some aspects of the topological space change and some other aspects remained fixed. The first invariants that appeared in mathematics are numbers and lines and area and volume of figures in terms of the dimensions. Some invariants are numerical, some are topological and some depend on differential structure. Those which are topological make differential topology and Those which depend on differential structure structure make differential geometry and the numerical ones make numerical Geometry.

§15. Differential geometry

The technique of differential calculus, integral calculus, linear algebra and multilinear algebra are use to study geometry objects and geometrical spaces in this field. It started by study of curves and spaces, which could be accessible to intuition and are generalized to higher dimensions and general manifolds. It is closely related to the field of differential equations that the techniques of algebra and analysis, which are both formal and verbal, are used to grasp geometric structures of object and spaces. One can also do differential geometry on infinite dimensional spaces, and of particular interests in differential geometry on the loop space, which was originated by Witten. To do the differential geometry on loop space or algebraic geometry on Hilbert scheme, or dynamical systems on the model in space of curves, or symplectic geometry on the space of symplectic curves is an approach to have achievements on moduli spaces and using Gerothendieck philosophy to do mathematics in families. In fact the concept of the space or a geometric object is a family of points. The family of lines and circles and other sub shapes of a space should be understood and addressed similarly. This is the art of modular mathematics. The idea of embedding geometrical space in another can be also formulated algebraically and that is the idea of family of representations.

§16. Representation Theory

Representation Theory is analogous to space embedding, therefore it is important to study moduli space of representations and also universal representations containing all information of the family of representations in questions.

These two approaches are geometric and algebraic in nature respectively. Both of these paths are pursued in 20th century. Considering the moduli space of representations means that he gets the idea of analogy between group representations and embedding of the spaces from geometry and then is studied with the same method of geometric thinking and intuition. But universal representation is an algebraic way to treat algebraic objects, although we are interested in the collection. We do not care in the algebraic approach if two representation are closed to each other in the moduli space. Algebraic method only cares about not losing information of any of the specific representations. In application, it becomes very handy if true representations are p-adically close to each other. The question is if we want to treat this as geometric or arithmetic nature. To my point of view, geometry has to do with R and intuition under the Shadow of the concept of length.

We do not wish to regard p-adic closeness as intuitive or graspable by our intuition. Some goes for general topological spaces, which are not luckily manifold. To ask topology is part of the algebraic treatments of mathematical objects and points.

§17. Analysis and limits

Analysis has two faces. One face towards inequalities and the other twards continuity and p-adic limits. Both of these faces are about algebraic treatments of algebraic objects together with the new concept of infinite sums and infinitesimals. Complex analysis and rigid spaces are also other faces of the above phenomena.

The order of real number is an important component in complex analysis and p-adic nature of rigid geometry is an important aspect of this theory although there are so much analogy with complex analysis and algebra geometry over complex numbers. This gives us the illusion that, maybe there exist such a thing as p-adic geometry and p-adic intuition and p-adic movement and p-adic time and P-adic space. As much as our mathematical models for physics understand p-adic time and p-adic space to make sense. And as far as our brain says geometric intuition is a function of the data that brain receives that all this together puts us in a position to say: p-adic geometry, why not? This has a truth in it and that is we shall not limit our cognition to highways of intuition known to humankind. There may always be new pathways for cognition, which have not been discovered before, or they have been discovered but not popularized. The method of calculus is one of these cognitive discoveries, which were popularized.

§18. Integral on manifolds

Integration on manifolds and solving differential equations on manifolds are one of the highest insights in pure mathematics. The former widely accepted and the latter not in its full potential yet.

There are people trying to do physics on manifolds. Both classical physics and string theory are studied in the framework of manifolds. But many differential equations, which are studied in the framework of modelling the natural phenomena, are not examined in the sitting of arbitrary manifolds. The concept of measure, topology, even metric are algebraic Concepts and forming the concept of topology and measure on a topological space could be sort of algebraic coordinatization of the space. To me deformation of algebraic structures and geometric structures both make sense. Therefore, one can think of the deformation of metric also as an algebraic method. Therefore the method of the deformation of metric used by Perelman to prove Thurston's geometrization conjecture. Geometry is pure intuition and calculation are all included in the algebraic realm. Is it seems at first glance that almost all mathematics is included in the realm of algebra. But in fact, the range of geometric intuition is much wider than algebraic formulations. Those who understand intuitive mathematics make algebraic formulations so that others are also able to do mathematics in their own range of cognition.

§19. Tangent bundles and Vector bundles

One can think of vector bundles as families of vector spaces. Among them tangent bundle is an intrinsic family of vector bundles associated to the deformation space. Same goes for fiber bundle which are family of fibers. But if deformation space is a symmetric space, we have intrinsic family of universal cover over the same deformation space. If deformation space is a moduli space, then also we have a universal space over the moduli space. In scheme theory algebra and geometry are united. One can not say that the method of base extension is algebraic or geometric. It can be interpreted in both languages. This is another instance of unification of the concepts of number and shape, which eventually lead to the field of arithmetic geometry. This was one more step towards Arakelov theory where the role of compactification of an arithmetic space is performed by analytical structures at infinity, which are in some sense geometric and in some sense analytic and hence algebraic. Differential geometry and differential topology could also be taught as unification of algebraic thinking and geometric intuition. This is return to original setup by Pythagoras.