On The Dialogue between Finite and Infinite

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Abstract: The mathematics in which the idea of infinity is present, we called infinite mathematics and the rest is called finite mathematics. In this paper, we study interactions between finite and infinite mathematics, and study the history of the development of the concept of infinity and its consequences in practice of mathematics.

Introduction

The idea of countable infinity or the set of natural numbers appeared very early in mathematics. For example, when Euclid proved that there are infinitely many primes, much before that they knew that there are infinitely many counting numbers, and as a consequence infinitely many rationals. But it took a long journey until the idea of uncountable infinity and continuous movements come into existence. To understand the concept of infinity, we will regard infinite mathematics as being more abstract against finite mathematics and we study incarnation and elevation among these two layers of abstractions in mathematics. From here the idea of dialogue between finite and infinite rises and leads us to the present study.

§1. Analogy and Interpretation

There is an analogy between finite and infinite mathematics but one can ask which one precedes the other. From human perspective finite mathematics is the origin and then by abstractization the idea of infinite mathematics became existeant. This is against our philosophy of science in which knowledge flows from depth torward the surface .in such a philosophy infinite mathematics preexists and finite mathematics comes afterward. Moving from finite to infinite by human mind is part of the path of truth from depth to surface and then from surface to depth. This is why in our philosophy of mathematics it make sense to speak of interpretation of finite mathematics in infinite mathematics. Because infinite is regarded as origin of the idea of finite. We do not insist on this philosophy here. That is why we consider finite and infinite mathematics as two different layers of abstractization of mathematics and only study the analogies between them. If an idea move from finite to infinite mathematics we call it elevation and if vice versa, we call it incarnation .This is because we believe infinite is more abstract than finite. Of course in history of mathematics there are many examples of elevation and only few examples of incarnation. Or at least this is how it seems to us; which seems against our philosophy of science.

§2. Incarnation

The concept of countable infinite or the set of natural numbers seems to be obtained by idealization from finite mathematics. So it is regarded as an elevated concept here. But there are a number of ideas which look like to be originated from development of infinite mathematics. The first example comes to our mind is the concept of limit. Although it was an ancient concept in the infinite world it incarnated in finite mathematics as a limit of finite structures. This could be inverse or direct limit which is an idea appearing in 20th century. The fact that a line has infinitely many points is very ancient but only cantor realized that this is different from infinite countable. The idea of continuity of movement of a point on line was first put forward by Iranian mathematician Khayyam. The idea of step by step discrete movement come from this origin but the development of finite mathematics appeared in late centuries. As a challenging question to our philosophy of science, one could ask, how could one think of natural numbers on a global infinite concept without starting from and building from local concepts of finite numbers? For example the concept of finite field could bring the concept of prime number. Infinite number of prime numbers could be thought of from finite and infinite perspective. But how could $\boldsymbol{N}$ exist without the experience of counting? The answer is a discrete model for semi-line. So, $\boldsymbol{Z}$ even existed before $\boldsymbol{N}$.

§ 3. Elevation

There are numerous ideas in finite and infinite mathematics which lead to analogous ideas in infinite mathematics. And there are many examples that an infinite object is characterized by finite information. For example, an infinite algebraic structure which is finitely generated, or can be represented as a limit of finite objects, are examples of elevation. These could both be considered as examples of elevation. Analogy between formulas in representations of finite groups and representation of compact groups was the first example that struck me. Then I got to know analogies between algebraic geometry over finite fields and algebraic geometry over R ***or C***. Riemann hypothesis could be easier understood over finite fields but first discovered over ***Q*** which is an infinite object! But then people try to generalize the finite methods of proving RH from finite fields to ***Q*** or other number fields. The fact is: finite mathematics is more accessible to our logical mind and infinite mathematics is more accessible to our intuitive mind. Interaction between finite and infinite could be thought of an interaction between intuition and logic. Elevation from finite to infinite mathematics could be thought as contribution of our logic to our intuition and incarnation as contribution of our intuition to our logic.

§ 4. Addition and Subtraction

These two arithmetic operations definitely originated in history as a part of finite mathematics, but at the second half of 19^{th} century, as the concept of algebraic structure became existent these arithmetic operations were to the global structure of a commutative group. Therefore, the concept of commutative group has arithmetic origin. At the end of story, the classification theorem of finitely generated abelian groups proves this claim. The concept of non-commutative group on the other hand, has its origin in the concept of global symmetry. This is also a global structure, but not based on the concept of integers ***Z***. Addition in non-commutative groups has roots in composition of symmetries. So you can think of it as a functional concept. The concept of function also has its roots in finite mathematics .But when this concept was officially introduced, it already took place as a part in infinite mathematics. Although the concept of symmetry originated from symmetries of objects. The group of symmetries could be an infinite object for its own sake. Maybe the simplest example of a non-commutative group which is infinite is the free group F_2 with two generators. But, what would be the finite analogue of a finitely generated free group.

§5. Multiplication

Multiplying numbers definitely originated as an idea in finite mathematics .Like addition and subtraction, it soon found continuous analogue and this was the beginning of philosophical controversies. People like Khayyam complained that product of two lengths is supposed to be surface area which is not of the same material type. Khayyam solved the problem by originating the concept of number as a ratio. It took a long time till the concept of ring was introduced.

Even before that the concept of field was discovered which was using the operation of division. The fact prime numbers are infinite was the first encounter of the concept of multiplication with infinite mathematics, which was of course pretty early in history of numbers. Addition of infinitely many numbers was also introduced very early. It was in Europe when multiplication of infinitely many numbers were discussed. (Is this claim really correct?) . Area of a region in plane is additive and Archemidus thought of infinitesimal areas, and that was the origin of the concept of infinite series, which appeared in calculus in 17th century. Analogue for multiplication was not available until the concept of logarithm appeared which translates infinite multiplication to infinite series. Infinite sums and infinite products didn't have analogues in finite mathematics.

§6.Division

Division for natural numbers is tied With Euclidean algorithm, which also has continuous analogue. The idea goes back to much earlier in history of mathematics. Egyptians used fractions of unity. Mixed numbers appeared much later. Division of arbitrary length was available after Thales but they needed to fix a length of unit before defining division. It became a global concept under the hands of Galois in 19th century. I guess that was the first time it entered infinite mathematics. The concept of all rational numbers (positive and negative) appeared after Descartes, and that was the first appearance of the concept of field materially. But the algebraic structure of a field appeared later. Division by zero was forbidden, and eventually it was accepted as a number called infinity (∞) , just unofficially as a simplifying model for understanding the limiting behaviors of functions on ***Z*** or ***R*.** The concept of real numbers and continuous movement should be thought as a part of infinite mathematics. I cannot imagine if the concept of infinity could be originated in finite mathematics independently. I believe that the concept of infinity has metaphysical origins. The first person who used line as a model for numbers was Khayyam. The concept of infinite line goes back to Thales, with some contributions and comments from Pythagoreans.

§7.Limit

As mentioned before, the concept of infinite came to existence independent of finite mathematics. But limit was a concept linking the two worlds of finite and infinite from the beginning. Archemidus was the initiator by considering infinite series. Pascal made contribution by claiming that the limiting cases of every theorem should be understood. The concept of limit of a point moving on a line without going to infinity appeared in calculus and origins of these ideas was in work of Fermat, were the concept of tangent to the curve appeared. Limit was an official part of calculus until Cauchy and Weierstrauss came up with the modern definition and made the concept rigorous. Limit of a sequence, limit of series of numbers and series of functions all became rigorous in 19th century. In 20th century, the idea of limit of points elevated to the ideas of limit of algebraic structures and limit of spaces. Also, p-adic numbers appeared as completions of ***Q*** by considering limit of rational numbers with respect to a norm. This was the first rigorous definition of real numbers also. Ring of integers of Q_p could also be considered as a limit of finite structures, which was the origin of the appearance of inverse limit.

§8. Inverse Limit

Inverse limit and direct limit are dual concepts. In inverse limit, we deal with extension and expansion of objects and in direct limit we deal with quotients and summaries of objects. This is a totally new concept of limit and originated new links between finite and infinite. Direct and inverse limit can be formulated in terms of universal properties, which extend the application of the concept of limit further. Of course universal properties come out of a holistic point of view to object, and although they make sense in finite mathematics and could be originated there, but they were born inside infinite mathematics. Union and intersection of sets and LCM and GCD of a set of numbers could very well be thought of as finite analogues of universal property. Intersection and union of a chain of sets would be similar to direct and inverse limit and universal property in the setting of set theory. It is easy to see ***R*** as the direct limit of some structure, but it is a challenge to see ***R*** as an inverse limit of structures. For p-adics the situation is more or less different. You can see Z_p as inverse or direct limit of structures easily. This is related to the fact that ***R*** has no rings of integers. Maybe this is the feature that it makes ***R*** interesting. $\overline{Q}$ Is an important example of direct limit and Gal ($\overline{Q} / Q$ ***)*** is an important example of inverse limit in arithmetic.

§9.Universal property

Universal properties make sense for both algebraic and geometric structures. It was not a long time in history for appearance of universal property in arithmetic until it appeared in geometry but it was a long philosophical path to treat geometric objects and space as algebraic structures. Many older concepts like intersection, union, direct sum, GCD, LCM, and others could be easily formulated of terms of universal properties. Definitely this new approach was a holistic view to many analytically defined objects in history of mathematics. This way definitely a move from finite to infinite mathematics. Because most infinite objects could be defined most naturally from holistic point of view. Direct limit and inverse limit particularly could be defined using universal properties. Although analytic definitions of direct and inverse limit were always available. These reformulations formed the conceptual background for a revolution in history of mathematics which happened under the influence of Grothendieck which showed that mathematics could be done holistically without reference to internal structure of objects and only in terms of morphisms between objects, with noticing the internal structure of morphisms. This approach made universal objects of critical importance in mathematics.

§10.Universal objects

To deal with universal objects using holistic perspective became a new art and industry in second half of 20th century. This an approach to deal with many infinite objects with a finite language. If one cannot grasp the definition of an infinite object with finite mathematics, it would be impossible to deal with that object in mathematical languages. Therefore it cannot be studied or even defined. This approach not only worked very well for arithmetic and algebraic objects, but also worked for geometric objects leading to a new perspective to moduli spaces. The most important and complex algebraic applications of universal objects I can think of in universal deformation ring appearing in Wiles' proof of Fermat's last theorem. In fact Wiles finds an isomorphism between two universal objects which are defined algebraically. He called this counting elliptic curves, which is an important philosophical comment because it bring analytical counting to a whole new perspective which is holistic. Doing geometry using moduli spaces was never initiated. But doing geometry on the moduli spaces became a new art. Of course moduli spaces could be defined analytically and holistically. The new holistic approach though was a step toward dealing with infinite geometric objects with finite language.

§11.Moduli spaces and compactification

One can think of a circle as moduli space of points with equal given distance from a given point. But the first time the concept of moduli spaces was introduced was when the projective space was defined as the moduli space of lines passing through a point in a three-dimensional space. This was a new approach to defining circle also. Circle would be thought of a moduli space of lines passing through origin in Cartesian plane. Projective space could be conceptualized as a homogenous compactification of plane beside stereographic projection. This idea comes from the concept of perspective in painting. This idea was introduced by Alberti, the teacher of Da Vinci. In fact the painting Mona Liza was so famous because it was one of the first paintings in which principle of perspective was respected. The next important moduli space was Grassmann space which was of similar nature. But moduli of elliptic curves with universal elliptic curve living above it and moduli of hyperbolic curves of given genus and moduli space of Abelian varieties of given dimension, were the first moduli spaces defined using universal property. Base change was an important feature of this moduli spaces. These moduli spaces have particular properties which is special to them. This is the subject of our study in the next section.

§12.hierarchy of moduli spaces

Projective space of dimension n can be represented as a union of C^i for i=0, 1,…,n. Also one could think of projective space as compactification of C^n which lower dimensional projective spaces. This feature is repeated in compactification of moduli spaces of curves of genus g, whose lower general moduli spaces could be used for compactification. Actually these compactification are not unique, but there is a minimal compactification which is not smooth unlike the prototype case of projective moduli space. The moduli space of Abelian varieties shares the same feature. Moduli space of Abelian varieties of dimension g is naturally compactified by moduli space of Abelian varieties of lower dimensions. There is a minimal compactification which is canonical but not smooth, and there are many non-canonical smooth compactifications which depend on combinatorial choices. In fact these compactifications and even the original moduli spaces can be thought of a schemes over ***Z.*** in fact, one needs some extrs structure to get schemes rather than stacks and that makes in take out some primes from Spec (***Z)***. Therefore, the moduli space eventually were translated to algebraic and arithmetic languages. These hierarchies can be used to define infinite dimensional Abelian varieties and curves of infinite genus. Let us start by history of appearance infinite dimensional objects.

§13.Infinite dimensional objects

Infinitely generated abelian groups and infinitely generated vector spaces were the first infinite dimensional objects which came into being. Probably this happened at the second half of 19th century. There are many infinite dimensional vector spaces which can be defined by finite languages, although they have infinite generators. For example some field extensions could be thought of infinite-dimensional vector spaces, like $\overline{Q}/Q$ or even R/Q . Then the analogy between finite and infinite come to play a role by considering analogy between the theory of finite dimensional vector spaces and infinite dimensional vector spaces to see what the common features are. Direct sums, quotients, and morphisms in general and many important theorems in the theory of infinite dimensional vector space, extend easily to infinite dimensional world. Functional analysis was pretty much originated to answer such questions. Axiom of choice could be an important ingredient in such a mathematics. The concept of norm on an infinite dimensional space could be easily axiomatized and that brings the question of a complex analogue for norms on vector spaces over ***C*** and this lead to the idea of Hilbert spaces which revolutionized the state of the art in infinite-dimensional case. This concept of course appeared at the beginning of 20th century by Hilbert!

§14.Hilbert spaces

The concept of Hilbert spaces generalize the methods of vector algebra and vector calculus to infinite dimensions. This is an irreplaceable tool in PDE, quantum mechanics, Fourier analysis and ergodic theory and also functional analysis. The important examples of Hilbert spaces are spaces of square integrable functions, space of sequences, Sobolev spaces, and Hardy spaces. Geometric initiation plays an important role in many aspects of Hilbert space theory. Exact analogues of Pythagorean Theorem and parallelogram law holds in Hilbert space. At a deeper level, perpendicular projections onto a subspace, plays a significant role in optimization problems and other aspects of a theory. There is a concept of orthonormal basis. Linear operations in Hilbert spaces are fairly concrete objects. Hilbert spaces are important in physics since they provide one of the best formulations of quantum mechanics. The observables are Hermitian operators on that space, the symmetries of the system are unitary operations, and measurements are orthogonal projection. The relationship between quantum mechanical symmetries and unitary operations provided an impetus for the development of unitary representations of groups initiated by Hermann Weyl. The spectral theory of continuous self-adjoint linear operations on a Hilbert space generalizes spectral decomposition of a matrix.

§15. Quadratic Forms

There are other important objects in finite mathematics beside linear vector space which deserve to be found in infinite framework. For example, there are many occasions in history of development of quadratic forms which ask for infinite-dimensional analogues. For example , the development of an infinite-dimensional theory of real or complex quadratic forms make sense, but not an infinite dimensional quadratic form with integer coefficients , since there are problem of convergence to be dealt with .

§ 16.Automorphic Forms

Does there exist a theory of automorphic representation of $GL(\infty, R)$ or infinite-dimensional analogues of other classical groups? Can we develop a Langlands program for such representations? How can one record a representation of $GL(\infty, R)$ n finite language? Some questions could be asked for representation of infinitely generated group. Infinitely generated Abelian group could be thought of an analogue infinite-dimensional vector spaces and infinitely generated non- Abelian groups could be thought of an analogue of infinite-dimensional classical groups.

§17.Infinite Dimensional Abelian Varieties

How does an infinite dimensional abelian variety make sense? At least, one could introduce it as a limit of a series of abelian varieties, each embedded in a higher dimensional variety. Or, as an infinite direct sum of infinite dimensional abelian varieties. Even an inverse limit of a series abelian varieties which can be arranged such that each abelian variety is image of a higher dimensional abelian variety in the same sequence. Of course, we expect that eventually one could talk about such infinite dimensional algebraic groups independently from finite mathematics, and with using limits and direct sums. Then the question of study of their moduli space come to consideration which is an infinite algebraic variety in its own right is very interesting to be studied. Then the question can be asked if there is an infinite dimensional theory of Siegel modular forms which fits the automorphic theory of representations of infinite dimensional classical groups? How can we try to write a q-expansion for such modular forms and so on?

§18. Curves of infinite Genus

Curves of infinite genus are more difficult to deal with than infinite dimensional abelian varieties. I can imagine an infinite sequence of Riemann surfaces each branched cover of a Riemann surface in the sequence of higher genera and then taking the inverse limit of such a sequence. Using modular curves such an infinite sequence of breached covers could be easily built and such inverse limit have already appeared in the history of modular forms. Next step is to develop the concept of Jacobian of such infinite genus curves and also studying the moduli space of such curves which is itself an infinite dimensional spaces. Does it make sense to develop a concept of infinite genera algebraic curve if the infinite dimensional moduli space is an algebraic space itself are both challenging questions. The rich theory of Riemann surfaces and differential geometry of them asks for generalization. There is a rich body of mathematics on the moduli space which also asks for generalization. For example does compactification of the infinite dimensional moduli space make sense? Can one study asymptotics of closed geodesics on this huge space? Can one think of a holomorphic structure on this moduli space and universal curve and if holomorphic sections of universal curves could be studied.

§19.Continuity and Limit

Contributions of the concept of infinity could be global or local. One of the most important local contributions is the concept of continuity. One may wonder that topological continuous functions makes sense in finite topological spaces. But we claim that point- set topology is a contribution of infinite mathematics to finite mathematics. The space of continuous functions on a space brings the concept of C*-algebras but this infinity is a global infinity rather than the local Infinitesimals, infinitely small phenomena we have in mind. Cauchy and Weierstrauss were the first who could grasp the concept of limit mathematically under a finite language. This procedure took 200 years. For two centuries the concept of limit was used only intuitively without a firm mathematical background. the concept of limit of points lead to limit of lines and circles and other geometric structures and eventually to the concept of inverse limit and direct limit which we explored in previous sections. All this can be thought of as a contribution of infinite mathematics to finite mathematics. The simplest being profinite groups with Galois group the most important example of them, which became important before the concept of profinite groups came to existence.

§20.Derivative and Integral and Infinite series

The concept of tangent line was an ancient concept but was considered as a limiting case of intersection of a line by curves under the hands of Fermat .Fermat related this concept to finding the maximum of a function. Newton and Leibniz developed this concept and created calculus. The concept of integral was considered as Archimedes calculated area of a part of parabola. Infinite series were also developed by him at the same point. Newton and Leibniz developed the concept of integral under influence of Archimedes and related it to derivative in the fundamental theorem of calculus. The first version of this theorem appeared in class notes of Isaac Barrow advisor of Newton. These notes were accessible to both Newton and Leibniz. Barrow brought together two ancient contributions of finite mathematics to infinite mathematics. After that differential equations and integral equations were the most important tools to model natural phenomena. That was when applied mathematics started to flourish. One shall note that derivative and integral have discrete analogues, which are studied in the theory of difference functions. This theory is still an infinite theory but its bounds with finite mathematics is more clear.

§21. Differential Equations and Integral Equations

Differential equations play a prominent in many disciplines including engineering, physics, economics and biology. If a self –contained formula for the solution is not available, the solution maybe numerically approximated using computers. Many different fields in mathematics and physics and engineering developed for study of particular differential equations. Functional analysis also contributed a lot to solution of differential equations. Qualitative study of solutions of differential equations lead to study dynamical systems. In differential equations one determines a global infinite system by finitely many quantitative data.

§22.Dynamical Systems

In dynamics, there are two main branches discrete and continuous dynamical systems. Continuous dynamical systems developed under the qualitative study of differential equations, but this theory brought up the idea that discrete dynamics of a self-map could also be studied. Finiteness results in arithmetic dynamics was one of the dialogues between finite and infinite. For finite mathematics we develop infinite mathematics and then try to look for finiteness in the infinite world we have created. Like the relation between dynamical systems on moduli space of curves and dynamics on curves themselves. In dynamical systems one determines the behavior of a global infinite system by finitely many qualitative data.

§23.Deformation of Geometries

Deformations of metrics on manifolds first approved as a feature in Einstein relativity. Then it appeared as a general philosophy of relative schemes in Grothendieck mathematics. Grothendieck believed that all geometric and algebraic Concepts should be studied in families except if they have discrete nature like arithmetic structures. Eventually these ideas lead to the formation of metric studied by Hamilton which eventually leads to proof of Poincare conjecture through Turston's suggestions by Perelman. The idea of deformation quantization was also studied by Connes and Kontsevich and collaborators. By Grothendieck philosophy one can even deform objects over finite field and sometimes, this leads to infinite objects.

§24.Conclusion

We see that finite mathematics and infinite mathematics are like two parallel streams which are correlated. Sometimes ideas developed in finite mathematics contributes to infinite mathematics and sometimes vice versa. But these influences opens and conquers the horizon of new connections between the concepts of finite and infinite.