On Holistic and Analytic Mathematics

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Abstract: Holistic and analytic are two of the important cognition types in educational psychology. In this paper, we investigate their implications on the way mathematicians do research and finally the borders they imply between holistic mathematics and analytic mathematics.

Introduction

In a previous paper, we discussed implications of pictorial, verbal and constructive cognition Styles on the nature of mathematicians performing their research and categorization of branches of mathematics to pictorial, verbal and constructive branches and contribution of these categories into each other. During this process, Holistic and analytic cognitive style naturally came into mind, and took part in categorization and how these categories interact with each other. In this paper, we will try to study holistic and analytic style independently of other cognition type and extrapolate on implications of these cognition types on the way mathematicians do mathematics and on the way in which mathematics is categorized under the Shadow of these cognitive types. At the end of paper, comes comments on the personality of holistic mathematician and analytic mathematicians which reminds us the work of Freeman Dyson “on birds and frogs”. Each of these two types of mathematicians have their own particular role in development of mathematics. These differences between cognition types definitely have implications on the type of mathematics these groups perform. We are going to comment on this aspect also.

On the cognitive type of holistic

Holistic mathematicians tend to be encyclopedic scientist. They move from whole towards parts and parts are defined for them under the shadow of whole.

This means that the personality of parts is defined based on the role they should play with respect to whole. A holistic mathematician does mathematics without paying attention to boundaries between different branches of mathematics which separates them. Therefore they have a global perspective towards mathematics. This personality makes them fit for being leaders in whole or in part. They could lead mathematicians in what the meaning of mathematics is, or lead them in research in a few connected fields or lead research in a particular field which is cultivated from a number of resources which live in neighboring fields. Holistic mathematicians are more interested in mathematical structures rather than mathematical objects and they care more about position rather than problem solving. They try to understand a mathematical situation by exploring the ways it could be generalized not by specializing to important and special cases. Universal objects and universal properties engage them more than particular objects or particular properties. Of course, there are different levels between parts and whole and each mathematician according to how much Holistic he is, chooses his role in development of new mathematics. It is important to mention that being holistic also effects on the way mathematician teach mathematics or study fellow mathematician's research.

On the cognitive type of analytic

Analytic mathematicians move from parts toward whole. Therefore the personality of whole for them is defined under the shadow of the personality of parts. These mathematicians have tendency to become expert in particular field or even particular methods of computation or particular tools to prove theorems. They may even be interested in special objects or special concepts and their research moves around these particular choices. These mathematicians like step by step proofs rather than proofs which require holistic intuition. They cannot see a whole branch of mathematics like a big picture. A branch of mathematics is like a branch of correlated theorems for them. It is the particular choices they make in doing mathematics which determines their approach to the whole. They have interests in particular periods of history of mathematics rather having an understanding of the whole. Therefor they live practically in a particular area in the past. Analytic mathematicians are the "frogs" in Freeman Dyson paper and holistic mathematicians are the “birds” who see things from above. Logic is an important tool of cognition for an analytic mathematician rather than intuition. Analytic mathematicians are at most leaders in a very particular field without paying attention to resources of ideas coming from other field. They lead a field of research trying to answer internal philosophical questions which are raised within the field.

On holistic algebra

Algebra is mainly verbal and analytic. But there are parts of algebra which are produced under holistic machineries. Example of holistic algebra, is parts of algebraic geometry developed by Grothendieck and collaborators and category theory developed by Eilenberg and MacLane. Of course, the function field formulation of algebraic geometry is also holistic. Grothendieck mathematics was very much influenced by Eilenberg and MacLane. This kind of algebra heavily influenced research in algebraic geometry and arithmetic geometry and eventually lead to solution of most of the historic problems in the field successfully. There are number of objects in number theory which are holistic. An example being the Galois groups and Galois representation which played an important role in development of the field in the last century and it continues to contribute as a computational tool which carries lots of information within itself.

Study of algebraic structures and associating global invariants like in algebraic number theory is another example of holistic algebra. (Co) homology theories are algebraic holistic invariants carrying a lot of information from the corresponding structure. Many of these ideas have roots in geometry which is mainly holistic. The concept of symmetry groups, Lie- groups and Lie-algebra are also parts of holistic algebra and should be understand from the perspective of whole to part philosophy.

On analytic algebra

Algebra is verbal and the mainstream of algebra is step by step and analytic. Constructing algebraic structures by generators and relations is an example of parts to whole Philosophy which is very appropriate to be translated to the language of computers. Computer calculations has contributed a lot to analytic algebra. Contributions of logic to algebra are also considered to be analytic. Because logic is both verbal and analytic. Analytic algebra is step-by-step and consist of local objects and invariants. Analytical algebra is philosophically independent, trying to answer questions of his own territory and not much relating to the resources of algebra, which is contrary to holistic algebra. An example of a holistic algebra is number theory which is almost entirely build on analogies between this field and other resources of ideas from outside the field. This nature of analytic algebra goes back to the cognitive structure of analytic cognition type of mathematicians doing this kind of algebra. Algebraic geometry and number theorists are usually holistic mathematician, but there is a big package of analytic objects like modular forms and L-function which became exceedingly important in these two fields. This is why there are difficult objects to be understood: mathematicians interested in them are cognitively holistic not analytic. This complicasy has been a serious burden to tackle the most difficult problems in arithmetic geometry.

On holistic analysis

L-functions and modular forms are examples of holistic objects in analysis. There are a number of holistic analytic objects in their neighbor like automorphic representations. This circle of ideas is known as Longlands conjectures, which are pioneering research plan in algebraic and analytic number theory. This, very much fits with the fact that number theorists are holistic. There is another sub-branch of number theory called analytic number theory, in which a number of results are holistic rather than analytic. In the phrase analytic number theory, analytic refers to its relations with analysis and real and complex numbers. For example, estimating of the growth of sequences are one of the holistic skills in analytic number theory. Functional analysis is also holistic in the sense that it deal with functions which are global objects, there is a particular style of doing functional analysis which is due to Grothendieck. This style is the holistic approach to deal with function spaces. Note that every time we deal with global objects in analysis is then we are doing holistic analysis. For example, the question of finding global solutions of differential equations or PDE's by moving from local solutions to Global solutions which is characterized as analytic rather than holistic. There are a number of results in complex analysis also which are holistic. Like the Riemann mapping theorem. They are also results in geometric analysis which are holistic.

On analytic analysis

Traditional analysis from Newton and Leibniz to Cauchy and Weierstrass is characterized to be analytic. Real and complex analysis, measure theory, differential equations, integral equations, Sequences and series, harmonic analysis, dynamical systems, and so on, are characterized as part of analytic analysis. Methods of proof are step-by-step and logical and theorems are local results. Functional equations and inequalities are parts of analysis which deal with global objects rather than local information. The field of inequalities officially started by Cauchy and then further developed in 19^{th} and 20^{th} Century. These are parts of their fields that can be studied at the level of high school also. This is important, because almost all analysis in high school tends to be holistic. Graphing polynomial functions or rational functions or other algebraic functions and other techniques related to them are holistic rather than analytic. This somehow shows that the mathematics taught in high school is biassed on the sense that, it is holistic and has no analytic components. This is not fair to students whose style of learning is analytic. The only local concept in the high school mathematics is the concept of limit which is not that analytic since one moves from global to local in understanding limits. So are all concepts in high school calculus. The curriculum of high school mathematics needs revision to include some analytic analysis as well.

On holistic and analytic combinatorics

Combinatorics is a discrete version of geometry and geometry is a continuous version of combinatorics. So by definition combinatorics is mostly holistic. Graph theory, discrete geometry and convex bodies are all holistic and about holistic objects and concepts.

Counting is one of the main Concepts in combinatorics and is used often in high school mathematics which is again holistic. Division of counting problem to several cases is highly intuitive and also holistic. Therefore, even the methods of proof are not analytic. Even applications of analysis to combinatorics are holistic. Functional analysis, dynamical systems, and estimating sequences are all holistic methods. So one asks, is there any room for analytic combinatorics? Is verbal combinatorics totally holistic? Is there any analytic verbal combinatorics?

In Kayley graphs of groups and geometric group Theory, objects are defined by equations which hold among generators and this is analytic approach in doing combinatorics of graphs. Still the methods of geometric group Theory are holistic rather than analytic. This points out to the fact that analytic combinatorics has never been studied, according to author's knowledge. Maybe this is because analysis deals with infinitesimals which are hard to define for finite or discrete objects.

On holistic and analytic geometry

Although discrete dynamical systems, convex and discrete geometry, and finite topological geometry are parts of combinatorial parts of mathematics, but they are all holistic. The continuous parts of the field geometry are partly holistic and partly analytic. Differential geometry for example, is analytic although there is local and global differential geometry. Algebraic topology is verbal but holistic, both in homotopy theory and in (co)homology theory. Manifolds and cell complexes are also holistic although analysis on manifold is local to global and therefore analytic. ODE and PDE on manifolds are also local to global and very similar to usual ODE and PDE. Geometric probability Theory is all analytic. The funny thing is that analytic geometry is part of algebraic geometry and mainly holistic. There is a sub-field of singularity theory which is local and therefore analytic. Physics is partly holistic and partly analytic. Holistic like mechanics and analytic like quantum theory. Even parts of mechanics are analytic like fluid mechanics. Optics and electromagnetic theory and wave theory in general are usually local to global and thus analytic. Statistical mechanics is also local to global and Therefore analytics not holistic. Analytic and holistic thinking are intertwined in geometry more than any other field of mathematics.

On the cognitive style of mathematics

I have never seen holistic mathematicians do analytic mathematics. Actually, by definition the only way to recognize if a mathematician is holistic is the mathematics they do. I have seen some mathematicians do both holistic and analytic mathematics but in any of these examples between being holistic or analytic, one of them is overcoming the other. One cannot be both moving from local to global and moving from global to local at the same time. Mind is not like double processor computers. One can start from local to global and then move from global to local which is analytic in my point of view like John Milnor. Or one can start from global to local and then move from local to global afterwards which is classified as holistic like Alexander Grothendieck. This is a very important point in mathematics education that students are taught both directions: from local to global and vice versa. It would be much better if the students are lead to a direction to study fields which are compatible to their cognitive structure. In fact many basic textbooks in mathematics is calculus, linear algebra, algebra, and analysis and combinatorics should be written for students in both cognitive Styles. Same is true for verbal, pictorial and kinetic styles of learning. I'm not sure though if one should mix these two styles of learning in writing textbooks or not.