Does Proof Bring Certainty?

or

On the Concepts of Doubt and Certainty

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Abstract. This article is about certainty in mathematics. We raise the question if proof brings certainty? What kind of proofs brings certainty? And if they don't, what is the replacement to achieve certainty in mathematics.

Introduction.

The concept of "certainty" is a central concept in mathematics. This is why Euclidean mathematics is considered so important. Euclid was the one who raised the standard of certainty in mathematics to a level that nurtured the life of mathematicians for thousands of years. Euclid's achievement was based on the concept of "proof", which in its own terms and conditions was based on the axiomatic philosophy of science introduced by Aristotle and the concept of proof made by Thales and Pythagoras. It is now time to reconsider standards of certainly, after 2000 years of absolute rule of the Pythagorean concept of proof. Does proof bring the ultimate certainty for us? Is it always the case for all types of proofs? Is there any replacement for the concept of proof to bring certainty for us? Does formal proofs and algebraic proofs bring more certainty than geometric proofs? Is geometric intuition is uncertain? Is the use of intuition against certainty? Is certainty a verbal concept which fits only the verbal mathematics? What is the pictorial analogue of certainty? How these two different conceptions of certainty compare? What are characteristics of a pictorial concept of certainty? What about doubt? Is doubt a verbal concept which fits only the verbal mathematics? What is the pictorial analogue of doubtfulness? How these two different conceptions of doubtfulness compare? In this paper, we explore different conceptions of doubt and certainty. We try to reconsider standards of Euclid in obtaining certainty, and have a wider conception of doubt than the Euclidean concept of doubt which is based on counterexamples.

Euclidean certainty

The Euclidean concept of certainty is based on the Euclidean concept of “proof” which is step by step, local to global, and analytic. In fact the Euclidean concept of proof is constructive. But this aspect has not fully survived through the history of mathematics. This was so much not the case that Brouwer set himself to revive the philosophy of constructive mathematics at the beginning of 20^{th} century. The standards of Euclid were propagated using his textbook “The Elements” which was the most well-read textbook after the scriptures. For thousands of years, mathematicians thought of “proof” as the ultimate standard of certainty, but this never limited them in their approach towards mathematics, meaning that the proof was never the ultimate tool for searching for the truth. In fact, argument was a final and finishing touch and there were almost always other sources for discovery of truth; for example, educated guess from some worked examples, generalization and intuition.

Euclidean doubt

In the standards of Euclid any mathematical understanding was in the form of a proposition which was always true or could be in general false. If it was true it should have been verified with a proof and if false it would be enough to find a “counterexample” to that proposition. This is certainly a particular approach to what mathematical knowledge consists of and all different approaches to make claim were accepted only as a philosophical statement which put forward some approaches on how to study the material but didn’t have mathematical value. The standards of proof and counterexamples were the only standards for propositions which were acceptable to build and develop the future path of mathematics. Even intuition was not acceptable as a final and terminating source of doubt for the truth of a proposition. It was so much the case that proving propositions which were counter-intuitive were considered fun and interesting and quite deep pieces of mathematics in the shadow of Euclidean concept of doubt. Certainly these concepts of certainty and doubt had roots in the verbal approach of Euclid to mathematics.

Verbal certainty

Verbal mathematics is based on analytic thinking and formal arguments and formal concepts. It is very much the Hilbertian mathematics or the Hilbertian view towards mathematics which protects the verbal approach towards mathematics in its full potential. This was based on an axiomatic approach to doing mathematics and leaning all the history of mathematics on the axioms supporting them. This point of view showed to be short in its access to the truth by ground breaking theorems of Godel. In fact, Godel showed that the concept of validity of a proposition cannot be interpreted as it representing an ultimate truth. Godel showed that verbal certainty brings only a relative cognition in the setting of axioms we started with. Meaning that, starting with a different set of axioms there may be other phenomena observable in mathematical nature. Still this observation did not destroy the imperial government of proof in the realm of certainty. Nor it changed our perspective towards the role of counterexamples in bring doubt to us towards the truth of a proposition. And certainly it did not change the official face of mathematics as a set of propositions which support our understanding of the mathematical nature.

Verbal doubt

Verbal doubt like the verbal certainty is based on analytic thinking and formal arguments and formal concepts. Failure of a single step in a setting of several steps in a row would be the single type of reason why a proposition fails. Sometimes one develops intuition from different ways a proposition fails or cannot be generalized and this could be thought as another source of certainty giving us a feeling that we know what we are talking about. But these two approaches confirm the previous concept of certainty. In this point of view there is a single conception of certainty against a single conception of doubt. We never face different types of certainty or different types of doubt, which are familiar phenomena while dealing with intuition and geometric thinking. In fact, verbal doubt and certainty can be understood having an algebraic nature against the geometric nature of pictorial doubt and certainty. Actually while doing geometry and topology the verbal standards of thought no longer help us to proceed in the realm of imagination to develop further mathematics.

Pictorial thought against verbal thought

Verbal thought is step by step, in the realm of order and time, local to global and analytic, where pictorial thought is holistic, global to local, outside the realm of time and sudden. Verbal thought and verbal mathematics is how we do algebra versus pictorial thought and pictorial mathematics is how we do topology. Ona can deforms objects in time in the realm of topology in a way not accessible in the realm of algebra. Deformation of algebraic structures can be conceived in a way not in the realm of time but in the realm of space and structure. The formal thought is geometry and topology can also be conceived but it doesn't replace the determining role of geometric intuition. Of course, intuitive thought can accept meaning in the realm of algebraic mathematics, the same way that formal arguments can be used in the realm of geometric and topological mathematics.

Pictorial certainty

Doing intuitive topology we face a different kind of certainty which has a completely different nature. The main feature of intuitive or pictorial certainty is that it consists of several layers of truth and each layer supports its own compatible concept of certainty. There is a hierarchy in these different levels or layers of certainty, some being more abstract and some being more material and there is give and take relationship between these kinds of certainty. Meaning that, there is a movement of cognition from above towards below, and another type of movement from below towards above. The geometric certainty move up and down in these layers, and the more abstract it becomes, the more unified it seems. As if, this way of thinking of certainty, pushes us up to a realm of unity of mathematics where the most abstract setting of concepts and relations makes the mathematical phenomena more observable. As if, we get closer to the realm of mathematical truth which manifests in lower layers of our mathematical understanding. This world of our mathematical knowledge is no longer a single layer of mathematical facts and can be handled with the Euclidean standards of doing mathematics. Euclidean certainty and Euclidean doubt are no longer appropriate concepts to deal with so many different layers of manifestations of mathematical truth. Therefore we have to develop another concept of mathematical doubt which has several layers of abstractness and therefore cannot possibly be based on the single concept of counterexamples.

Pictorial doubt

Pictorial doubt makes sense against pictorial certainty. Therefore, there must be several layers of doubt conceivable for several layers of abstractness, we are dealing with. If fact, in each layer of abstractness there is a single concept of validity which makes one to doubt or feel certain in this realm. Therefore, the language of mathematical propositions can no longer be the single language in which mathematical phenomena are being introduced. There could be several manifestations of mathematical truth. But for now, we recognize a few layers of languages of mathematical settings: the layers of formal objects; the layer of concepts; the layer of deformation of objects; the layer of moduli spaces; the layer of mathematical structures; the layers of axioms and ultimately the layer of mathematical truth.

Verbal certainty against pictorial certainty

The first difference coming into mind is the rigidity of the concept of verbal certainty against the flexibility of the concept of pictorial certainty. One can zoom in and zoom out to find an appropriate concept of certainty for any piece of mathematical knowledge in the pictorial setting. While in the verbal setting, a single concept of rigor is supposed to handle all the spectrum of truth, which looks very much limiting and hand tying.

Verbal doubt against pictorial doubt

Verbal doubt is a rejecting final argument or counter example but pictorial doubt is the beginning of deep investigation. Asking questions like: what is the right level of abstractness to formulate the mathematical question? What are the borders of truth? What are the relevant concepts of certainty? What are the correct counter examples limiting the truth? What are the right generalizations which can be formulated? What are the correct perspectives and correct mathematical languages which suit for thinking about the problem? Rigidity of verbal doubt is against further understanding and flexibility of pictorial doubt is a gateway to further knowledge. But what do we expect from doubt and certainty? Do we need them just to develop further knowledge or for higher standards of thinking? Is pictorial certainty also capable of raising our standards of thinking?

What do we expect from certainty?

Having a vision of what is going on, so that you can predict the mathematical truth and the truth of mathematical statements, without having to prove with all details is holistic point of view towards the truth which could be the fruit of certainty. Of course, only proving things, is not enough for finding such a perspective. Many proofs are not introducing a good picture of what is really going on. Understanding the mathematical truth is like discovering a new city or a new neighborhood. Having understood some parts, does not necessarily tell you what is going on in a neighboring area. But this is what we expect from certainty. Also we don't want an image from the truth which breaks when having more knowledge. So, approximating and simplifying intuitions are not supporting certainty. We need to have a picture of what is going on which keeps holding true for a long time after discovery. This limits the concept of certainty when we are dealing with a changing system or structure, since we don't know in which direction the change is occurring. In short, even if proof brings certainty in particular occasions, it is by far the most expensive means of getting access to certainty. There are cheaper and more effective ways to achieve as much certainty as can be afforded using proofs.

What are the sources of doubt in modern mathematics?

How does a modern mathematician feel like, when he has doubts and he is not certain about the structure or the mathematical situation he is exploring? What are the sources of his doubts? If he doesn't know how to generalize the statement, or if he finds special cases which could have been understood more explicitly, or if what he has found is somehow against intuition, or if the picture given does not match with the picture given by other statements, or if one feels like there is more structure into the material he is exploring, and the question is asking for further investigation. Doubt is the source of further investigation and seeking a deeper understanding rather than the reason to reject the truth of a statement. This brings us to the realm of considering mathematician's satisfaction. The mathematician feels certain if he feels he is satisfied with the research. This calls for asking about the sources of dissatisfaction of a mathematician after making a mathematical discovery. What makes the mathematician satisfied and what keeps him away from satisfaction?

What are the standards of satisfaction in modern mathematics?

May be the reader asks if the question of satisfaction is the same as the question of certainty. Does a mathematician feels satisfied when he is certain enough about the truth of his findings? Certainly, the certainty is a source of satisfaction. On the other hand, how satisfied you are about understanding the mathematical situation you are facing with has something to do with the depth of certainty furnishing your mathematical experience. This is a sort of an argument why speaking of satisfaction is relevant in the context of certainty.

When does a mathematical research call for further investigation? How does one find more satisfaction by understanding and learning about a mathematical situation? When there is a feeling of something deeper being hidden inside and that is a matter of your perspective towards the beauty and of course a matter of vision and intuition. It is a totally intuitive matter if you are satisfied by the understanding you have gained about the mathematical situation or not. Therefore certainty is a matter of completely intuitive nature. Same is satisfaction and certainty you gain from proof. Why should we limit ourselves to the method of "proof" in finding satisfaction about our experience of learning about mathematical truth.