A Mathematical Framework for Mathematical Theories

Arash Rastegar
Sharif University of Technology

We intend to systematically develop a mathematical formalism in which concepts and their relations, flexibility of objects in representing conceptual background, different layers of logical implications, theorems and their logical and conceptual relevance, and dictionaries between different mathematical theories is taken into consideration. Next step would be plugging in human aspects of theory development, like skills of theoricians in formulating appropriate assumptions and research goals and superposition of concept mappings.

Introduction

Category theory gives us a new perspective towards mathematical objects. One particular aspect of this theory is introduction of the possibility of characterization of mathematical objects by their external properties. For example, one can uniquely determine many objects by the set of structure preserving maps from these objects to others, without taking the internal construction of these objects into consideration. This gives us an abstract setting in which one can bring in concepts like functors and their representability. This sheds new light on our understanding of mathematical theories. But still when we compare this formalism with the most basic building of mathematical formalisms, namely the book of ‘Elements`, one immediately recognises the weakness of this formalism in grasping the mathematical beauty and surprising depth of numerical and geometrical structures appearing in this ancient document of human intellect.

In this paper, we intend to develop a mathematical formalism which grasps a wider meaning for objects and morphisms between them and enables us to bring in the world of background concepts and their relations. A fruit of this extension would be different layers of logical implications, and new meanings for ‘theorems` and their logical and conceptual relevance. This may give us chance to understand dictionaries between seemingly unrelated mathematical theories in a more general framework.

Objects and assumptions on objects

There could be different types of objects in a theory. In classical Euclidean geometry we have points, lines, circles, ellipses, parabolas and hyperbolas. There could be different types of morphisms between these different types of objects. For example, one can define morphisms between points and other types of objects by inclusion, morphisms between objects of the same type by congruence, and morphisms between conic sections by projective transformations.

One can define new types of objects by diagrams of different types of objects and morphisms between them. Examples of these new types of objects are a pair of lines, or a pair of a conic with a point on it, or a triple of a conic with a line tangent to it at a point on the conic.

Defining such new objects, one can get new types of morphisms between these complexes of objects. For example, one can define a ‘triangle` by combining three lines and points and a particular arrangement of their morphisms.

Then one can define morphisms between these complexes of objects by a set of congruence between objects of the same type which are compatible with morphisms of inclusion.

The notion of ‘sub-object, could relate two different types of objects. For example, a point could be a sub-object of a line and of a plane. Objects in a theory are usually sub-objects of a universal object. For example, all objects in Euclidean geometry are sub-objects of the Euclidean space, or free finitely generated abelian groups are sub-groups of a universal group which is not finitely generated, or Galois representations to general linear group over finite fields could be thought of as sub-objects of a universal Galois representation to general linear group over a universal ring. Note that a sub-object is not necessarily a sub-set of its super-object.

Also, relative objects could be introduced using morphisms between objects, for example a point of a line makes a relative object. Objects could share sub-objects. For example, a pair of lines could share a point on them, or a pair of groups could share a sub-group. One way to define morphisms between objects and relative-objects is when some types of objects are canonically associated to some other types of objects. For example, to a pair of a point on a conic one can associate a pair of a point and a line tangent to conic on that point.

One could define a ‘family` of objects by considering all sub-objects of a given object. The family of all subgroups of a given group and the family of all lines on Euclidean plane are examples of families. If there are natural metric topologies on a family of objects one can make a precise meaning of a sequence f objects tending to another in the limit. Even a family of objects of special type could tend to other types of objects in the limit. Like, circles tending to a point or hyperbolas tending to a pair of common axis lines.

One could limit a family to a ‘sub-family` by assuming one or several assumptions on sub-objects of a given object. For example, the set of all lines in the Euclidean space passing through a fixed point forms a family of lines, and the set of all number-fields as sub-objects of the complex field which are finite over rational numbers also form a family.

Concepts and concept mappings

Concepts concern a family of objects or relative objects which could be of different types. Concepts are in fact equivalence relations on family of objects which are compatible with the language and intuition of human mind. In fact, concepts categorize families of objects by defining natural equivalence relations on them. For example, consider the family of pairs of line in Euclidean plane. Then the concept of parallelism would be an equivalence relation on this family. Note that a pair of equal lines could be regarded as a third equivalence class or as an element of any of the two classes of lines. All these choices are natural concepts to define.

Concepts are related if the corresponding equivalence relations are related by inclusion. More precisely, one could extend a family to a bigger family in such a way that the concept on the extended family induces the sub-concept on the original family. One can represent such a relation symbolically by an arrow. This makes the notion of concept mapping more precise. For example the concept of parallelism for the family of pairs of lines in Euclidian plane is related to the concept of parallelism in Euclidean space for the extended family. So, Concepts could be specialized and generalized to sub-families and super-families of objects. If one allows families of sub-objects be compared to a family of objects, then one has an extended version of conceptual relationship. For example, the the notion of parallelism

'Sub-concepts` could also be introduced by comparison of equivalence relations. We say that an equivalence relation is included in another if each equivalence class in the latter is union of equivalence classes of the former. For example, the concept of parallelism with three equivalence classes in a sub-concept of both of the other two concepts of parallelism with two equivalence classes. Two concepts could have a common sub-concept or a common super-concept.

One could define intersection and union of concepts to be the largest common sub-concept and smallest common super-concept. If smallest common super-concept of two given concepts is trivial, we say that these concepts are 'independent'.

Morphisms between concepts and relative concepts could be introduced by defining morphisms between equivalence relations. To define such a morphism one should start from a morphism between families of objects defined by association and try to make it compatible with the equivalence relations corresponding to concepts on families on both sides.

A family of concepts is defined to be the collection of all sub-concepts of a given concept which are of given type. A family of concepts could be specialized to a smaller sub-family by assuming certain assumptions. For example a family of all sub-concepts containing a given concept as a sub-concept could be defined.

Sometimes, families of objects could be summarized in universal objects. This could lead us to the intuition that 'families of objects could also be thought of as objects`. It is important to define a natural topology on a family of objects. This will lead to the notion of open concepts in which objects in one of equivalence classes can not tend to objects in other equivalence classes in the limit.

Concepts could be of different types. They could also tend to other types of concepts in the limit! For example, if we consider pairs of lines in a plane as a family of objects, being parallel or overlapping or equal is a concept on this family. Equal or non-equal is a sub-concept, and lines of particular relative angles or parallel is a super-concept. The super-concept could be thought as of different type of the other two. Indeed, the super concept is a discrete concept while, in other first two concepts, one could tend in the limit to other equivalence classes of concepts.

One can think of concepts as objects and define 'higher concepts` by fixing equivalence relations on families of concepts. One can define theories in which always concepts can be thought of as objects. There could even be theories where in addition to the previous property, objects are always concepts for lower objects. We call such theories non-Noetherian. There are in fact different layers of abstractness in such theories, in which, the words 'object` and 'concept` could have different meanings.

Fixing any type of objects or a collection of such types, and also fixing any particular object of given type or a collection of objects of given different types, one can associate families of objects and on each family one can associate a collection of background concepts. One could arrange morphisms between objects and also arrange morphisms between associated families, which so that they induce morphisms between background concepts. So, different types of objects could be related by morphisms and different concepts could also be related by morphisms. Therefore, one can consider compatible families of objects and background concepts and their types. For example fixing a hyperbola, one can consider families of points on the hyperbola or families of lines tangent to the hyperbola. The two connected components of the hyperbola introduce a concept on each of these families. Association of the tangent line to a point on the conic introduces a morphism between families compatible with background concepts.

Universal objects and their background concepts

Fixing a universal object one can induce families of objects using this universal object. Concepts on such families are called background concepts for this universal family. For example, parallelism is a background concept for the universal object of plane where the associated family is the family of pairs of lines on the plane, and being full (surjective) is a background concept for the universal Galois representation where the associated family is the family of induced residual representations.

To understand background concepts of a universal object, one should understand all the concept relations and concept mapping related to sub-objects of the universal object. For example, a Euclidean line accepts points as its sub-objects, and the only way one could speak of concepts on this set points is to introduce equivalence relations on the which could not be nontrivial, by the symmetry of the line. If we choose a pair of a point and the universal line as a sub-object, then fixing a sub-object would let us introduce concepts on this family, and this would make room for concepts of closed and open sub-sets of the real-line, and ordinal structures. This would make the concept-map of the Euclidean line more complicated.

Evidently, concept-map of the Euclidean plane is more complicated than that of Euclidean line. The reason is that, line is a sub-object of a plane and sub-objects of sub-objects of a universal object could be thought of as sub-objects of the universal object. Another example is the concept map of an irreducible residual representation which is simpler than the concept map of the original universal representation. An irreducible residual representation has no sub-representation, but we can consider representations of sub-groups and related sub-objects as sub-objects of the residual representation.

Sometimes, we limit ourselves to objects with simpler background concept-map to avoid complexity, but in that case, the complexity is pushed towards relations between these objects. For example, when we define a notion of irreducibility and reduce everything to irreducible situation, we are trying to avoid complexity of objects. On the other hand, sometimes, we prefer to push complexity towards the objects we use and prefer to have simpler relations between these objects. One shall be careful about this when trying to define a dictionary between two different theories, if they are compatible in this respect.

Logical implications and concept relations

Particular objects should not be important but their type should be under focus. We could define objects of the same type to be objects which are isomorphic, but this would be a limitation of our formalism, because then we would have to think about internal structure of objects. For example, in Euclidean geometry hyperbolas could be thought as of the same type, however they could be non-congruent. Considering objects up to isomorphisms would make our formalism very similar to category theory.

Logical implication could mean inclusion or surjection of families of objects, or in general morphisms between families of objects which are compatible with concepts or even types of concepts in consideration. It is evident how an injection induces a logical implication. When we have surjection between families of objects, concepts on the second family induce concepts on the first family and surjection gives a logical implication. A general logical implication can break into surjection and injections and trivial implications of sub-concepts.

Sometimes, morphisms between families are induced by morphisms between universal objects and thus logical implications could be embodied in morphisms between universal objects, which are compatible with some background concepts of these universal objects. For example, two line parallel in Euclidean plane, remain parallel in a Euclidean space containing it, is such a logical implication.

One could introduce types of logical implications, sub-implications and super-implications by considering a logical implication as an object. This way it is not anymore the case that all propositions are as valid logically, but there would be different types of validity. For example, an implication in Euclidean geometry could be true over the complex numbers or only over the real numbers. This way one can see that many propositions previously regarded as surprising are no longer surprising! Propositions are combinations of natural implications and in order to generalize a proposition one should be aware of the types of logical implications, so that one can decide if they could also be generalized. In this formalism, one could prove characterization results on the ways in which a proposition could be generalized.

One can consider 'families` of logical implications. By families of logical implications we mean all sub-implications of a given implication which are of given type. Therefore, families of propositions make sense. If we regard logical implications as objects, the notion of concept could be translated to the language of logical implications. Indeed, we call an equivalence relation on a family of logical implications a 'wisdom`. Notions of sub-wisdom, super-wisdom and morphisms between wisdoms could be introduced.

Theorems and phenomena

Theorems and propositions consist of a chain of logical implications or a chain of families of logical implications. One can introduce types of theorems according to types of logical implications. Being aware of the types of logical implications is extremely crucial when trying to generalize a theorem. By generalizing a theorem, we mean extending the families of logical implications or an isomorphic model to super-families. To define isomorphisms between logical implications, one should define morphisms between logical implications. To make this possible, one shall think of logical implications as objects and define morphisms of logical implications as morphisms between these complicated objects. It is reasonable to assume that such morphisms respect the corresponding wisdoms on the initial and terminal theories.

Phenomena consist of a collection of logical implications which could be used in different layers of abstractness. More precisely, a phenomenon is an isomorphism class of logical implications. A mathematical phenomenon is in fact a theme unifying a package of logical implications in different layers of abstractness. For example, the first theorem of isomorphism can be thought as a phenomenon. It is interesting if non-trivial families of phenomena which preserve types are found.

A 'mathematical formalism, consists of a package of correlated phenomena which could be used to understand certain mathematical structures. More precisely, a mathematical formalism consists of an isomorphism class of theorems. Linear algebra is an example of a mathematical formalism.

Morphisms between mathematical theories

A theory consists of a set of objects and families of sub-objects and their types, concepts and their types and logical implications and theorems. One could define morphisms between theories taking objects to concepts and types of objects to types of concepts. Therefore, in different layers of abstractness, one could find isomorphic theories.

Morphisms of level zero between theories which takes objects to objects and background concepts to background concepts also take phenomena to phenomena and are called dictionaries. Morphisms between theories commute with morphisms between logical implications and take mathematical formalisms to mathematical formalisms. For example, taking points to circles of fixed radius and lines to strips of the same fixed width, would give a morphism from Euclidean geometry into itself, which takes concepts to concepts and preserves their types. This morphism induces an isomorphism of Euclidean geometry formalism into itself.

One can extend a theory to a bigger one by extending objects to larger families of objects and generalizing concepts together with their types in a way that they are compatible to the old settings. It is handy to assume that all objects in a theory should be sub-objects of a global object. This way, it would be easy to handle the notion of sub-theory. A sub-family of objects together with induced concepts and their types would form a sub-theory. This way, one can consider relative theories and define morphism between relative theories. For example, Euclidean geometry of plane is a sub-theory of Euclidean geometry of three-dimensional space.

To form quotient theories, one can summarize similar objects in a group of objects in a way compatible to types of objects and types of concepts. More precisely, it is enough to define quotients of families of objects of different types by a sub-family. This could be defined as an equivalence relation on the ambient family of objects for which each equivalence class consists of family of objects and each equivalence class intersects the sub-family in a single object. This could be done in a way compatible with concepts and their types. This naturally defines quotients concepts and quotient types of concepts. It is evident that quotient theories of a given theory are not necessarily isomorphic theories. Note that this notion of quotient differs from the one used in algebraic structures.

If there is another sub-theory which could rise to the same quotient, we call it a section, as we define sections of morphisms between theories. One can consider sections of quotient theories as 'families` of theories. The set of all sub-theories of a given theory also could be thought as a family of theories. This notion could be limited by some further assumptions to extend the notion of families of theories.

Automorphisms of mathematical theories

One can define endomorphisms of a theory as morphisms from a theory to itself. They are allowed to change types of objects and concepts. Automorphisms could fix the objects but change morphisms between them. Automorphisms of theories are endomorphisms which define a self-correspondence between objects and concepts which are one to one and are compatible with their types. For example, in projective plane, taking lines to points, and point to lines of the dual space, is an automorphism of the geometry of projective plane.

One can also define endomorphisms and automorphisms of relative theories. One defines relative endomorphisms and relative automorphisms of relative theories being identity on the sub-theories. An automorphism gives a one-to-one correspondence between logical implications. It is interesting to understand dynamics of automorphisms if we could define a topology on logical implications. An automorphism of degree two is called a duality. An automorphism of higher finite degree is called ‘torsion` symmetry. As usual, one is allowed to combine automorphisms. Automorphisms of a theory form a group. One can define quotient of a theory by an automorphism, or by a group of automorphisms.

The quotient of a theory by a group of automorphisms is always a theory. One could also ask if the part of the theory fixed by a group of automorphisms is a sub-theory. It seems that this is not necessarily the case, and one can limit the concepts of automorphism to such. To have invariant theory plausible one shall limit automorphism theory to an appropriate theory.

Suppose that one has developed an appropriate invariant theory for mathematical theories, one could try to prove a Galois-theory for embeddings of theories. A one-to one correspondence between sub-groups of an automorphism group, and the set of sub-theories of a relative theory could be arranged. This way, the problem of understanding sub-theories of a relative theory could be translated to understanding sub-groups of a given group. It is interesting to see if the notion of normal sub-group translates to sub-theories of relative theories as it happens in the theory of Galois.