A Conjecture on the Concept of Generalization

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Abstract: We try to give several examples and supporting arguments for a philosophical conjecture about joining two or several generalizations of a theorem in a common theorem which accepts all those generalizations as special cases. We briefly discuss the philosophical background of the above speculation.

Introduction

The main example we have in mind is the celebrated Riemann-Roch theorem. Hirzebruch generalized it to varieties of arbitrary dimension which is nowadays called Hirzebruch-Riemann-Roch (HRR). Two different generalizations of this version of HRR theorem are introduced. One is the Grothendick-Riemann-Roch (GRR) which uses the language of K-theory and the other one is the Atiyah-Singer index theorem (ASIT) which uses the language of differential operators. At first look it does not seem at all possible to find a common generalization of GRR and ASIT which means to combine the language of two theorems, which one is algebraic and the other is very analytic. In this paper, we will try to give supporting philosophical arguments for the conjecture that it is indeed possible to join GRR and ASIT in a common theorem.

In fact, we conjecture that any two generalizations of a common theorem can be joined in a common theorem specializing to both.

On the concept of specialization

Many mathematicians who are analytic try to understand a piece of mathematical content by specializing to an important particularity which contains all features of the mathematical truth under speculation. To find such a special case which sheds light on the general case is an art. Because, it should be chosen in a way that one can effectively use particularity of the situation. For example, many algebraic geometers use toric varieties as a computable special case to check to truth of their general conjectures. In the process of finding specializations, there appears a curious phenomena. Sometimes one can find two independent specializations that contain two different features of the general results. In fact, there are examples of theorems with several independent specializations. Sometimes these specializations can be reformulated in a language which makes it difficult to see why they are particularity of the general theorem in question, and sometimes several such reformulation are possible. This gives us the picture of an upside down tree where the trunk is the general theorem and the branches are specializations. Then one may wonder if such a picture could be given for generalizations of a given specialty? In other words, could the specialty be considered as the trunk and generalizations as the branches? The aim of this paper, is to give philosophical reasons why it is not the case. Any two generalizations of a given specialty can be joined.

On the concept of generalization

Holistic mathematicians try to learn by generalizations. This means that, when they try to understand a piece of mathematics, they think about ways in which it can be generalized. Sometimes two generalizations seem independent. As if different features of the original theorem are generalized in each of them. Sometimes the special case has to be reformulated so that it accepts a generalization. It could be that two different reformulations lead to two different generalizations. This gives us the picture that maybe different generalizations are sometimes independent because the language they are formulated is seems not related to each other. Sometimes theorems hold true in limiting cases which could be formulated in a different framework. This limiting facts, should not be thought as specializations, but the theorem together with the limiting cases should be taught as a generalization of the original theorem. Sometimes the theorem itself is discovered to be the limiting case of a family of mathematical situations. This is also the case where the big family together with the limiting case should be considered as a generalization of the original theorem. One could imagine that a theorem in two different formulations is the limit of two different mathematical facts in two different families and one can hardly imagine why these two independent generalizations could come from a single common source. But in this paper, we are going to claim that it is indeed the case.

On theorems having several generalizations

Having generalizations and a specializations several scenarios come into mind. For example a generalization of a theorem could have several specialization which don't look like the original theorem at all. Also, it is a curious question if it is possible for a theorem to have more than two generalizations at the same time. Or is it the case that if you look carefully each branching is doubling and several branches are the results of several doubling. I think in nature it is the case that one of the branches is the older one and other one have been grown on the older branch and sometimes two such sub branches are closer to each other and eventually they seem be as strong as the original branch and it looks as if there has been trisection but it is really not the case originally. If this be the case in mathematics also we will have to consider the case of double branching only and understand what happens philosophically. Then the concept of several generalizations could be understood one by one and treated separately. This means that, at the moment we should not worry about a theorem having several generalizations at the same time. Because, if we look at the tree of generalization in the base of time, we realize that only double branching has happened in the history of mathematics. Although it may look different at the end.

On independent generalizations

For two generalizations of the same theorem which have two different formulations and languages, it looks very hard to be joined by a common generalization. This is not the only challenge. There could be that the two generations are specialized in two different meanings of the world to the particular theorem. For example, one could be a particularity, and one could be a limiting case. Also one generalization could be leading to two independent specializations which could be thought as a common generalization to all special cases which brings back the picture of upside down tree to our mind. Thus one could find a generalization of several known special cases at once. This gives us the picture that the more one generalizes the more one moves towards unification and this is against the believe that two independent generalizations are possible. This point of view of more abstractness is coexistent with more unity is a very Platonic idea. As if the mathematical ideas are flowing down from a common source and they get multiplied and lose their unity while coming down. This is exactly the picture of an upside down tree. But why it is the case that any two independent generalizations could be joined in a common generalization? The Platonic idea is very holistic and does not satisfy analytic mathematicians, which need a local reason why branching is happening only from above.

On joining two generalization

Let us start with a single mathematical idea which has two seemingly independent generalization, and suppose we have been successful in finding a common generalization to these two independent general cases. Now there are several comments coming to our mind. One is that this picture does not look like an upside down tree. Since moving from above to below it seems that two branches have joined each other again which does not happen in a tree. But philosophically it looks relevant for a general theorem, to have two independent specialization which has a common specialization in a common language or after a reformulation. This is more similar to a flow of water in a river where branches could be formed and then there could be joins again. Then one asks why not like a river it be the case that several branches joined together and unified and then division to branches happen and sometimes the branches even joined each other? These speculations at least have one important consequence and that is if joining two generalizations is always possible, this is by no means a local phenomena. Maybe two different generations after several steps of even more generations are ready to be joined together. The picture of the river could also be replaced by the picture of blood circulation. There is a unique source like the heart or the Seas and there is a direction in circulation of mathematical truth. This contains both pictures of upside down branching and usual branching.

On joining several generalizations

Having the picture of blood circulation in body or water circulation on earth in mind which are more global ways to look at generalization and specialization, we see that our conjecture of joining generalizations could cease to survive if you believe in a single accumulation of mathematical truth and existence of a flow of mathematical truth or a kind of movement to and from the single source. But these two ideas are already present in the Platonic philosophy of mathematics and we don't have to come up with a modern formulation which fits this framework. This places us in the mental situation that joining independent generalizations is maybe always possible but it is by no means an immediate fact. But on the other hand, joining several generalizations of a common phenomena could be a very good motivation of, and good source of, ideas for making new mathematics. So, one can not say that believing in such a conjecture is redundant. But it could be a very long term mathematical project to join two independent generalizations. This means that although we expect GRR and ASIT join each other in a single framework, but this is by no means an immediate and trivial thing to do. It may take several layers of generalization on both sides and several years of cooperation of analysts and algebraists, to get to the point of joining the two mathematical ideas.

A philosophical conjecture

Let us reformulate and repeat our philosophical conjecture again at the end of our discussion.

Conjecture A: Any theorem which has two independent generalizations eventually admits a common generalization to them.

Conjecture B: Any two theorems which have the same particular cases or limiting cases, could be joined in a common generalization to them both.

In fact, conjecture A and conjecture B are the same philosophical facts as is evident from the speculations we went through, but in practice they look different because sometimes the common special case is discovered after the discovery of the two theorems and sometimes before that. This order, makes the difference between conjecture A and contractor B.

But the purpose of the two conjectures and the philosophical background and claim of the two conjectures is the same truth. This common truth is existence of a realm of unity for mathematics which has several other consequences. We will discuss other consequences in future papers.