On Assessment of Problem Solving

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We discuss a few different approaches to the question of how we manage to solve problems and introduce a mathematical model for the process of problem solving. Then we use our model in order to formulate an assessment strategy which has positive improving effect on development of students' skills and their ability of solving problems.

Introduction

The way we think of the process of problem solving is directly influenced by our philosophy of thinking, which in turn is influenced by the way we think of human being. The traditional approach to this question limits us to considering concepts and their relations. In this setting atlas of concepts would give us an appropriate terminology for mathematical modeling of the process of thinking [١]. The more recent approaches consider not only concepts but also mathematical skills needed for problem solving. In this setting atlas of concepts and atlas of skills would be a nice package for modeling the process of mathematical discovery and problem solving. After developing assessments according to these models, we go further and introduce an assessment strategy based on a new model for human thinking, in which several different correlated layers of abstractness are considered. Skills and concepts are only two of these layers which we could think of them as analogues of language and thought. Language and thought are of different levels of abstractness which are related by generalization in one direction and symbolization in the other direction. Some other examples of other layers of abstractness will be introduced.

Modeling problem solving via concepts

Limiting the language of thought into concepts and their relations has the advantage of having available a very simple model of learning and thus of the process of discovery and problem solving, namely, atlas of concepts. The theory of atlas of concepts is an extended version of concept map which is flexible for modeling creativity.

In this framework, a problem solver tries to find a concept map which fits the problem better than other options and then works on extending the concept map by new concepts which are useful to the solution of the problem. The new formed concept mapping is stored as a part of the atlas of concepts to be used later on, for solving similar problems.

One can formulate different aspects of problem solving in terms of concepts and their relations. For example, creativity can be thought of, in terms of concepts. Any problem introduces a configuration of relations between previous concepts. Not all previous relations appear in the setting of the problem. In order to have the previous relations hold again in this setting, one introduces new concepts which connect the old concepts in the manner we expected from other maps in the atlas. This is what we call creativity. So in this model, a creative mind creates concepts and relations between concepts.

In the model of atlas of concepts, problem solving has the following stages:

Stage ١-Student translates the problem to the language of concepts and their relations.

Stage ٢-Student compares the concept map to atlas of concepts and finds possible relations between these concepts which are relevant to the problem.

Stage ٣-Student uses the old concepts or creates new concepts to form the expected relationships between already recognized concepts.

Stage ٤-Student recognizes new relationships between concepts in the concept map.

Stage ٥-Student goes through the steps ٢ to ٤ over and over until the concept map is extended enough to solve the problem.

Stage ٦-Student summarizes the concept map to the subset which covers the solution.

Stage ٧-Student translates the solution from the language of concepts and their relations to the language of the problem.

Modeling problem solving via concepts and skills

Students have different abilities in working with concepts. Having a rich atlas of concepts would not lead to perfection in problem solving. Students have different levels of abilities in learning from their atlas of concepts and in general, their mathematical experiences. The theory of atlas of skills is an appropriate language for mathematical modeling of students' ability of working with concepts [٢]. Atlas of skills is a collection of skills formed by pre-skills which could join in different manners to form different skills. Atlas of concepts together with atlas of skills form a nice package for modeling the process of problem solving. Definitely a more skillful student would do better facing a problem both in terms of solving the problem and learning from the experience. On the other hand, it is difficult to understand thinking in terms of the concepts themselves. But it is possible to understand the associated symbols and their relations. The language of skills is related to these symbols stronger than it is related to the abstract concepts and thus gives us a more appropriate ground for assessing the process of problem solving in terms of concrete evidence.

It is possible to write about skills of problem solving in terms of concepts and their relations and independent of the particular mathematical content of the problem. For example, one could introduce different levels of skillfulness in each of the stages of problem solving in the language of atlas of concepts. Here is a suggestion:

Stage ١-Student translates the problem to the language of concepts and their relations.

Level ١-Student recognizes the concepts relevant to the problem.

Level ٢-Student recognizes the relations between the concepts relevant to the problem.

Level ٣-Student is able to translate the assumptions of the problem to the language of concepts.

Level ٤-Student is able to translate the problem completely to the language of concepts.

Stage ٢-Student compares the concept map to atlas of concepts and finds possible relations between these concepts which are relevant to the problem.

Level ١-Student is able to find each of the concepts in his/her atlas of concepts.

Level ٢-Student is able to find a concept map in his/her atlas of concepts which has the same concepts as the problem.

Level ٣-Student compares the relations holding between the concepts in atlas and the relations between the concepts relevant to the problem.

Level ٤-Student chooses a concept map in his atlas best fitting to the concepts and their relations in the problem.

Stage ٣-Student uses the old concepts or creates new concepts to form the expected relationships between already recognized concepts.

Level ١-Student recognizes concepts which are related in the atlas but not in the setting of the problem.

Level ٢- Student is able to relate these concepts using other concepts in the atlas compatible to the setting of the problem.

Level ٣-Student is able to relate these concepts by creating new concepts.

Level ٤-Student is able to relate these concepts by creating new concepts which are relevant to the setting of the problem.

Stage ٤-Student recognizes new relationships between concepts in the concept map.
Level ١-Student finds relations between concepts using the concepts in the atlas.
Level ٢-Student relates a new concept to the old concepts.
Level ٣-Student relates the new concepts to each other.
Level ٤-Student relates the concepts in direction of the solution of the problem.
Stage ٥-Going through the steps ٢ to ٤ over and over until the concept map is extended enough to solve the problem.
Level ١-Student is able to extend the concept map of the problem.
Level ٢-Student extends the concept map in direction of the solution of the problem.
Level ٣-Student makes a decision whether he/she has to go through steps ٢ to ٤ again.
Level ٤-Students recognizes when the extended concept map is enough to solve the problem.
Stage ٦-Student summarizes the concept map to the subset which covers the solution.
Level ١-Student recognizes the portion of the map needed for solution of the problem.
Level ٢-Student is able to rewrite the concept map as a series of maps in which new concepts are relating the old concepts to form the old relations between concepts.
Level ٣-Student is able to summarize the new maps in the atlas.
Level ٤-Student is able to summarize the new maps in the atlas in direction of simplifying the solution.
Stage ٧-Student translates the solution from the language of concepts and their relations to the language of the problem.
Level ١-Student is able to translate the new concepts to the language of the problem.
Level ٢-Student is able to translate the new relations between concepts to the language of the problem.
Level ٣-Student is able to translate the solution to the language of the problem.
Level ٤-Student is able to translate the process of solving the problem to the language of the problem.

Having all these different levels of sophistication on hand, with no mention of the mathematical content of the problem, it shall be mentioned that our assessment would be more accurate if we could organize these levels according to the particular mathematical content of each problem.

The question of understanding how a problem is solved or how a particular occasion of creativity comes into existence is still untouched. We believe that, our model for human thinking has been too simple to be able to explain how we solve problems. Therefore, the next thing to do is upgrading our view towards the process of thinking.

A new model for human thinking

A realistic view towards the process of discovery shall emphasize two major functions of the human mind, namely, symbolization and generalization. We think of a problem in different languages which are of different levels of abstractness. We translate these languages to more (resp. less) abstract ones by generalization (resp. symbolization) of concepts.

We generalize the mathematical content, in order to see the connections with other areas related to the problem, and we symbolize the content in order to perform calculations and computations. There is no unique way of symbolizing the concepts and their relations. Therefore, a concept map in some level of abstractness specializes to a few concept maps in a lower level and itself is a specialization of a concept map from a higher level. Thus, in the process of thinking, not only students extend atlas of concepts in each level, but also translate different levels of abstractness to each other to see the connections and perform calculations. Here is a suggestion for a correlated system of languages of different levels of abstractness:

Language ١- The language of formulas and symbols.

Language ٢- The language of mathematical concepts and their relations.

Language ٣- The language of growth and deformation of mathematical concepts.

Language ٤- The language of deformation space which is the ambient space for concepts' growth and deformation.

Language ٥- The language of the logical system which is relevant to the problem.

Language ٦- The language of connection which exists between our mind and logical system of mathematics.

Language ٧- The language of creation of mathematical systems which are isomorphic to the logical system of the problem.

For example, we have a symbol for the concept of a triangle: Three non-collinear points A, B, C in the plain and the fragments connecting them. Then there is the concept of a triangle and then we can continuously move a triangle to deform it to particular triangles. Then we have the moduli space of triangles as the deformation space. Euclidean geometry is the mathematical system beyond and its postulates are connecting our mind to this logical system. Different models of Euclidean and non-Euclidean geometry are the mathematical systems which are created to gain a better understanding of the concept of a triangle. One can see that the notion of a triangle is discussed from different abstract points of view. Generalizations and symbolizations between these levels do help us understand the problem better. If we consider these different languages in our assessment, we have a better understanding of the mind of the mind of student in the process of problem solving.

Suggestions for assessment of problem solving

A mathematical model for the process of thinking would not be of any significance, except if it is able to give concrete norms for assessment of students' skills in problem solving. Here is a suggestion for different levels of students' sophistication in each of the languages of thought introduced above:

Language ١- The language of formulas and symbols.

Level ٠- Student is not able to work with formulas and symbols, and is not familiar with the rules governing them.

Level ١- Student has translated the problem to the language of formulas and symbols, but is not able to compute or make an argument.

Level ٢- Student has solved the problem accurately after translating it to mathematical symbols.

Level ٣- Student is able to translate the solution to the language of the problem.

Level ٤- Students has considered all the concepts behind every step of calculations and arguments.

Language ٢- The language of mathematical concepts and their relations.

Level ٠- Student has considered the concepts and the relations behind calculations and arguments and the relations between symbols.

Level ١- The process of problem solving is affected by the relations discovered between the relevant concepts.

Level ٢- Student has chosen key concepts and has solved the problem by considering their relation to other concepts relevant to the problem.

Level ٣- Student has considered the relations between concepts, during the process of problem solving and after the problem is solved.
Level ٤- Student has solved the problem and understands the problem and solution in terms of concepts and their relations and independent of formulas and symbols.
Language ٣- The language of growth and deformation of mathematical concepts.
Level ٠- After solving the problem some of the concepts are grown and understood better.
Level ١- Student has considered the relations between the extended concepts and other concepts relevant to the problem.
Level ٢- Student has considered the new relations through the process of problem solving, after the problem is solved.
Level ٣- After solution of the problem, some concepts are extended so much that they have absorbed other concepts as a special case.
Level ٤- Student has compared the relations between concepts after and before the solution.
Language ٤- The language of deformation space which is the ambient space for concepts' growth and deformation.
Level ٠- Student has interfered the process of growth and deformation of concepts.
Level ١- Student has affected the growth and deformation of concepts in direction of solving the problem.
Level ٢- Student understands the limitations in growth and deformation of concepts.
Level ٣- Student has taken advantage of the relations of concepts in growth and deforming particular concepts.
Level ٤- Student successfully guides growth of concepts in a way that it absorbs a pre-destined concept as a special case.
Language ٥- The language of the logical system which is relevant to the problem.
Level ٠- Students recognizes a system isomorphic but different from the mathematical system of the problem.
Level ١- Student is able to translate the data from this system to the system of the problem.
Level ٢- Student uses the translation from one logical system to the other in solution of the problem.
Level ٣- Student accurately translates the problem to an isomorphism logical system.

Level ٤- Student understands the logical system of the problem independent of concepts and their relations.

Language ٦- The language of connection which exists between our mind and logical system of mathematics.

Level ٠- Student is able to reconstruct a logical system in his/her mind which is similar to the logical system of the problem.

Level ١- Student directly understands the logical system in his/her mind in a way that is useful in solution of the problem.

Level ٢- Student has taken role in reconstruction of the logical system in his/her mind.

Level ٣- Student has directed the process of reconstruction of the logical system in his/her mind in a way that is useful to the solution of the problem.

Level٤- The reconstruction of the logical system by student leads to the solution.

Language ٧- The language of creation of mathematical systems which are isomorphic to the logical system of the problem.

Level ٠- Student has expanded the logical system of the problem.

Level ١- Student has connected the logical system of the problem to other mathematical system in order to expand it.

Level ٢- Expansion of the system is lead to generalization of the problem.

Level ٣- Student has compared the expanded system with other mathematical systems and related the generalization of the problem to similar problems in other systems.

Level ٤- Student has created a new mathematical system to solve the problem.

The affect of assessment on student skills

This assessment document could be used both as a self-assessment for students and an assessment conducted by the teacher. As a document for self-assessment, there is the possibility that the complicated language of the document be misunderstood by the students. It would be an almost impossible task to explain them to the student in full detail. On the other hand, the teacher has to ask many questions from the student, in order to be able to assess his/her level of sophistication in problem solving. It is possible though, to get information from the portfolio of the student or student’s written exam, and assess student’s abilities in a way that is referable to some extent. The question is what would be the effect of such an assessment on student’s progress in problem solving. This could be the subject of a field study. What seems evident is that this assessment document is trying to help students to do simple mathematics the way mathematicians do it.

References

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